Integer-Valued Polynomials

LA Math Circle
High School II
Dillon Zhi

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1 Introduction

Some polynomials take integer values $p(x)$ for all integers x. The obvious examples are the ones where all the coefficients are integers. But the coefficient don't all have to be integers.

Examples:

In this session, you will find out which polynomials are integer-valued polynomials.

The main tool we will use is **finite differences**. For any expression $f(x)$ with x as a variable, we will define the **first difference** $\Delta_x f(x)$ by

$$\Delta_x f(x) = f(x+1) - f(x).$$

Usually, we'll just write "Δ" instead of "Δ_x". To put it another way, if f is a function, we can write Δf to mean the function that takes x and returns $f(x+1) - f(x)$.

Example:

x	x^2	$\Delta(x^2)$	$\Delta(\Delta(x^2))$	$\Delta(\Delta(\Delta(x^2)))$
0	0	1		
1	1	3		
2	4			
3				
4				
5				
6				
7				

Instead of $\Delta(\Delta f)$, we can also just write $\Delta^2 f$, the **second difference** of f. More generally, Δ^r means, "take the first difference r times" (that is, the r-th difference).

2 Finite difference basics

1. Compute and simplify the following first differences:

    a. $\Delta(x^2)$

    b. $\Delta(x^3)$

    c. $\Delta(7^x)$

    d. $\Delta(x!)$

2. a. The quantity $\Delta(f(x) + g(x))$ is equal to $\Delta f(x) + \Delta g(x)$, since both are equal to ______________________________ after expanding.

    b. The quantity $\Delta(cf(x))$ is equal to $c\Delta f(x)$, where c is a constant: both are equal to ____________________.

3. a. Let f be a function which takes integer inputs and returns real outputs. (We usually write this as "$f : \mathbb{Z} \to \mathbb{R}$".) If f is a constant function, then we have $\Delta f =$ _____ everywhere.
    Show that the converse of this also holds.

    b. Now suppose that $f : \mathbb{R} \to \mathbb{R}$. Give an example of a function f such that $\Delta f = 0$ but f is not a constant function.

c. Continuing from the previous question, show that f must be a constant function if we also assume that f is a polynomial function.

d. Prove that if $f(x)$ and $g(x)$ are polynomials such that $\Delta f(x) = \Delta g(x)$, then there is a constant C such that $f(x) = g(x) + C$.

Also, if f and g are integer-valued functions, then C is an integer.

4. Recall that the **degree** of a polynomial in x is the highest power of x which has a non-zero coefficient. Constant polynomials have degree 0, *except* the zero polynomial, which is a special case: we will leave the degree of the zero polynomial undefined (it's often said to be $-\infty$).

   Prove that if $p(x)$ is a polynomial of degree $n \geq 1$, then $\Delta p(x)$ is a polynomial of degree $n - 1$.

5. If you take the n-th difference of a degree n polynomial $p(x)$, then you'll end up with __________________________.

3 The special polynomials $P_n(x)$

We will now focus on the polynomials

$$P_n(x) = \frac{1}{n!} x(x-1)(x-2)(x-3)\cdots(x-n+1).$$

For instance, $P_3(x) = x(x-1)(x-2)/6$, $P_2(x) = x(x-1)/2$, $P_1(x) = x$, and $P_0(x) = 1$ (think about that last one for a little bit).

1. What is $\Delta P_n(x)$, for each nonnegative integer n?

2. Prove that $P_n(x)$ is an integer whenever x is an integer.

3. Remember the principle of mathematical **induction**: Suppose a statement is true for $n = 0$. Also suppose that if the statement were true for a particular n, then it must also be true for $n + 1$. Then we can conclude that the statement is true for all integers $n \geq 0$.

   Prove (by induction) that every integer-valued polynomial $p(x)$ is an "integer linear combination" of $P_0(x), P_1(x), ...$ That is,

   $$p(x) = c_n P_n(x) + c_{n-1} P_{n-1}(x) + \cdots + c_1 P_1(x) + c_0 P_0(x)$$

   for some integers $c_n, c_{n-1}, ..., c_1, c_0$.

   Hint:

   a. *Prove that the statement is true if $p(x)$ is a constant polynomial.*
   b. *Assume that the statement is true for all polynomials of degree n, for some particular n. Prove that the statement is true for all polynomials of degree $n + 1$.*

4. Prove that every real polynomial $p(x)$ is a real linear combination of $P_0(x), P_1(x), \ldots$. That is,

$$p(x) = c_n P_n(x) + c_{n-1} P_{n-1}(x) + \cdots + c_1 P_1(x) + c_0 P_0(x)$$

for some real numbers $c_n, c_{n-1}, \ldots, c_1, c_0$.

5. We now know that every integer-valued polynomial $p(x)$ looks like

$$p(x) = c_n P_n(x) + c_{n-1} P_{n-1}(x) + \cdots + c_1 P_1(x) + c_0 P_0(x)$$

for some integers $c_n, c_{n-1}, \ldots, c_1, c_0$. But is it possible for $p(x)$ to be expressed in this form in more than one way? We will soon see.

   a. Let $c_n, c_{n-1}, \ldots, c_1, c_0$ be real numbers. Prove that if

   $$c_n P_n(x) + c_{n-1} P_{n-1}(x) + \cdots + c_1 P_1(x) + c_0 P_0(x) = 0$$

   for all x, then $c_n, c_{n-1}, \ldots, c_1, c_0$ must all be 0.

   b. Prove that every polynomial can be expressed in the form

   $$c_n P_n(x) + c_{n-1} P_{n-1}(x) + \cdots + c_1 P_1(x) + c_0 P_0(x)$$

   in only one way.

4 Computing the coefficients

You've proven that every integer-valued polynomial can be expressed as a unique integer-linear combination of $P_0(x), P_1(x), ...,$ but how do you compute the actual values $c_0, c_1, ...$?

1. Let's say you are told that $p(x)$ is a polynomial of degree n, and you are given the values of $p(0), p(1), ..., p(n)$, which are integers. Come up with an algorithm to find the integers $c_0, c_1, ..., c_{n-1}, c_n$.

2. Demonstrate your algorithm on this example, where $p(x)$ is a polynomial of degree 3:

x	$p(x)$	$\Delta(p(x))$	$\Delta^2(p(x))$	$\Delta^3(p(x))$
0	2			
1	-3			
2	0			
3	5			

5 Bonus problems

1. Find all integer-valued polynomials in two variables. That is, the value of the polynomial $p(x, y)$ is an integer whenever x and y are both integers.

2. Find all integer-valued rational functions with rational coefficients. (A "rational function with rational coefficients" is a function f which can be expressed in the form $f(x) = A(x)/B(x)$, where both $A(x)$ and $B(x)$ are polynomials with rational coefficients.)