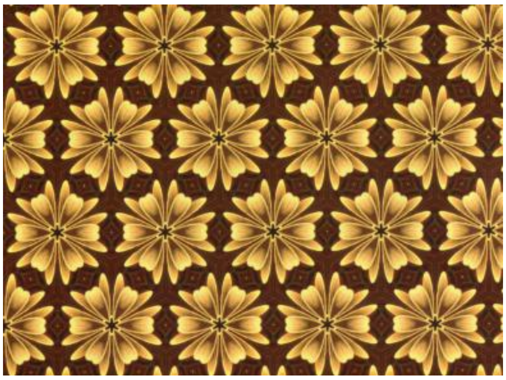
Wallpaper Patterns

We will develop a signature[1] to classify wallpaper patterns. A wallpaper pattern is a pattern in the plane that has translational symmetry in two distinct directions, so that that it fills up the whole plane. We also require that the group of symmetries be countable, to exclude things like a blank wallpaper.

§1 4 Types of Wallpaper Symmetries

A **mirror** symmetry is reflection about a line. Its signature is $*$. An integer n following $*$ denotes n-fold mirror symmetry, the intersection of n mirror lines. Two intersections of mirror lines are considered the same if we can perform a translation and rotation that sends one to the other, while leaving the pattern the same. There are various possible combinations of mirror symmetries. This flower pattern has signature $*632$: there are three distinct point of intersecting mirror lines with 6, 3, and 2 mirror lines respectively.



Exercise 1.1. Design a wallpaper pattern with signature $*2222$

Another symmetry is n-fold **rotational** symmetry about a point, whose signature is written $\mathbf{n}$. Multiple bold numbers means multiple points of rotational symmetry. Two points of rotational symmetry are considered the same if we can perform a translation + rotation sending one to the other, while leaving the pattern the same.

There are also patterns with both kinds of symmetries. To classify such patterns, first find all the mirror symmetries, then all the rotational symmetries that are *not accounted for by the mirror symmetries.* By convention we write the rotational symmetries before the $*$.

[1]also called "orbifold notation," introduced by Thurston and popularized by John Conway

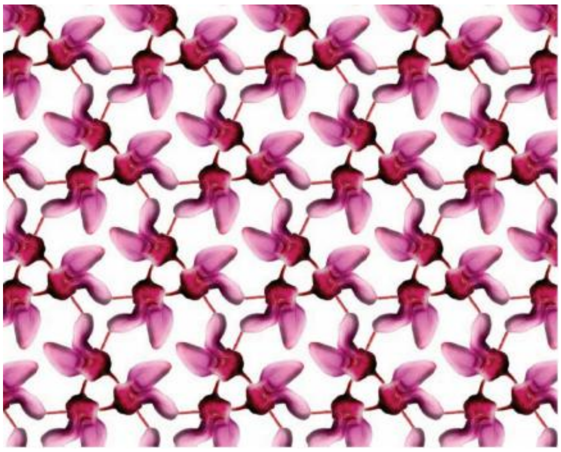


Figure 1: Signature **333**.

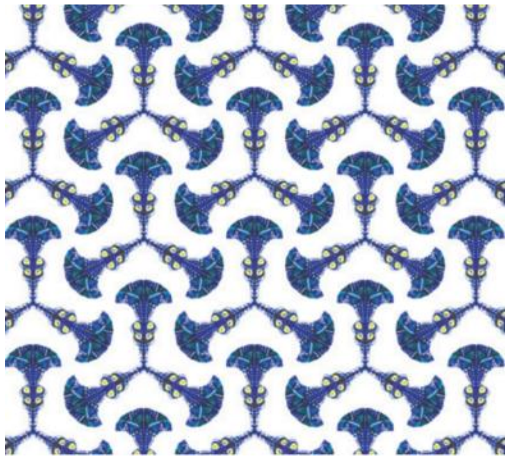


Figure 2: Mirror and rotation symmetry

Problem 1.2. Mark the three rotation points in Figure 1.

Problem 1.3. Find the signature of the pattern in Figure 2. $\mathbf{3} * 3$

Some exceptional cases: It is possible to have two *different* parallel mirror lines. In this situation the signature is $**$.

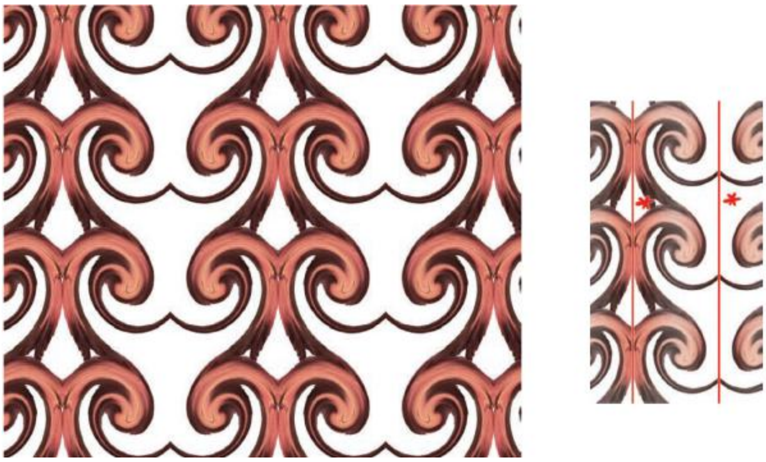


Figure 3: Check that the mirror lines really are different!

Exercise 1.4. Draw another wallpaper pattern with signature $**$.

There are two other types of symmetries. The first called a **miracle** whose signature is written $\times$. It is the result of a *glide reflection*, which is translation along a line followed by reflection about that line.

This occurs when there is orientation-reversing symmetry not accounted for by a mirror. For example, if we modify Figure 3 slightly we get a signature of $*\times$.

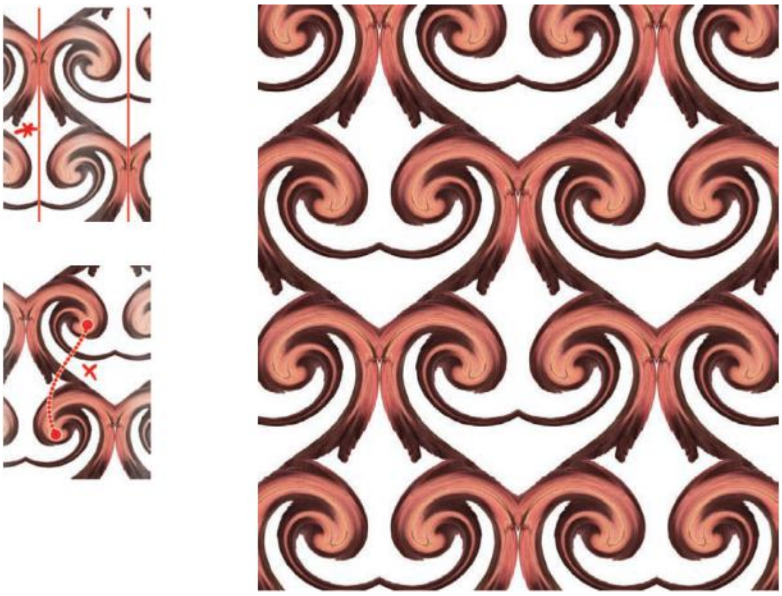


Figure 4: Signature $*\times$. There is a glide reflection (shown by the by the dotted line) taking the clockwise spiral to the counter-clockwise spiral, reversing orientation.

Problem 1.5. Find the signatures of these ancient patterns: $*\times$ and $4*2$

Figure 5: A pattern from the Alhambra in Spain

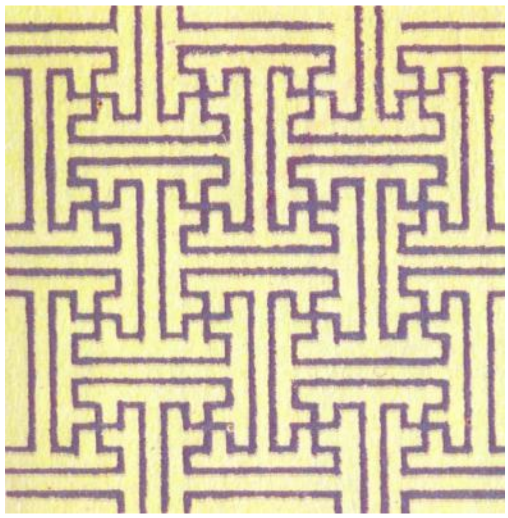


Figure 6: A porcelain pattern from China

There is another exceptional case with two miracles, where there are two glide reflection symmetries along distinct lines. There are other glide reflections, but they can be obtained by composing the two marked in the diagram.

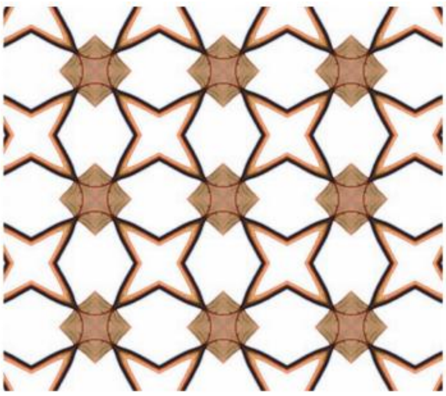
Figure 7: There are two distinct mirrorless crossings, so the signature is $\times\times$.

Lastly, if none of the above symmetries appear in the pattern, then there is only regular translational symmetry, which we denote by $\bigcirc$.

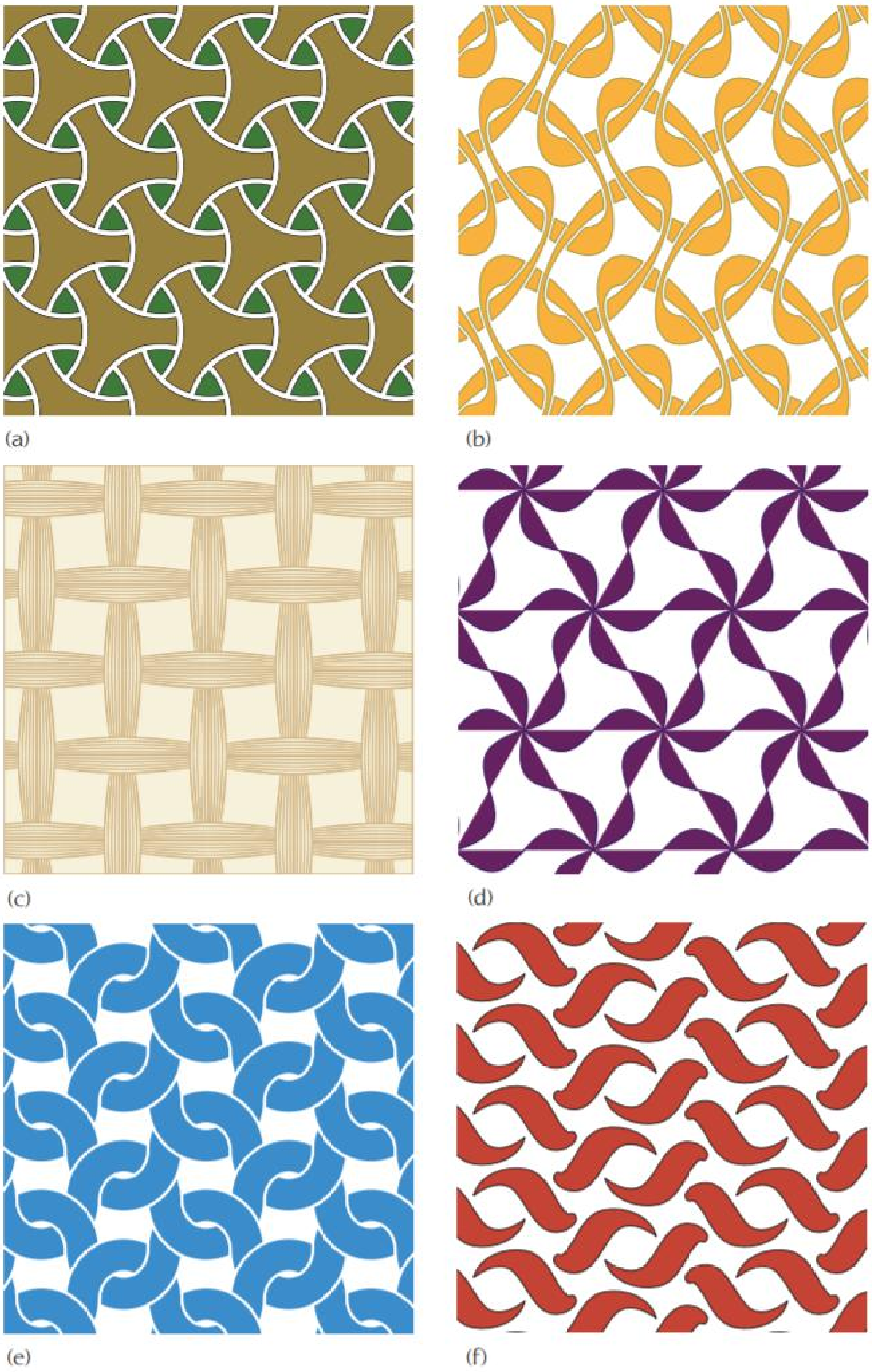
In summary, to find the signature of a pattern:

1. Find the mirror lines $(*)$ and the distinct intersections
2. Find the rotational points of symmetry not account for by reflections.
3. Look for any miracles $(\times)$ i.e. glide reflections that do not cross a mirror line.
4. If you found none of the above, it is just $\bigcirc$.

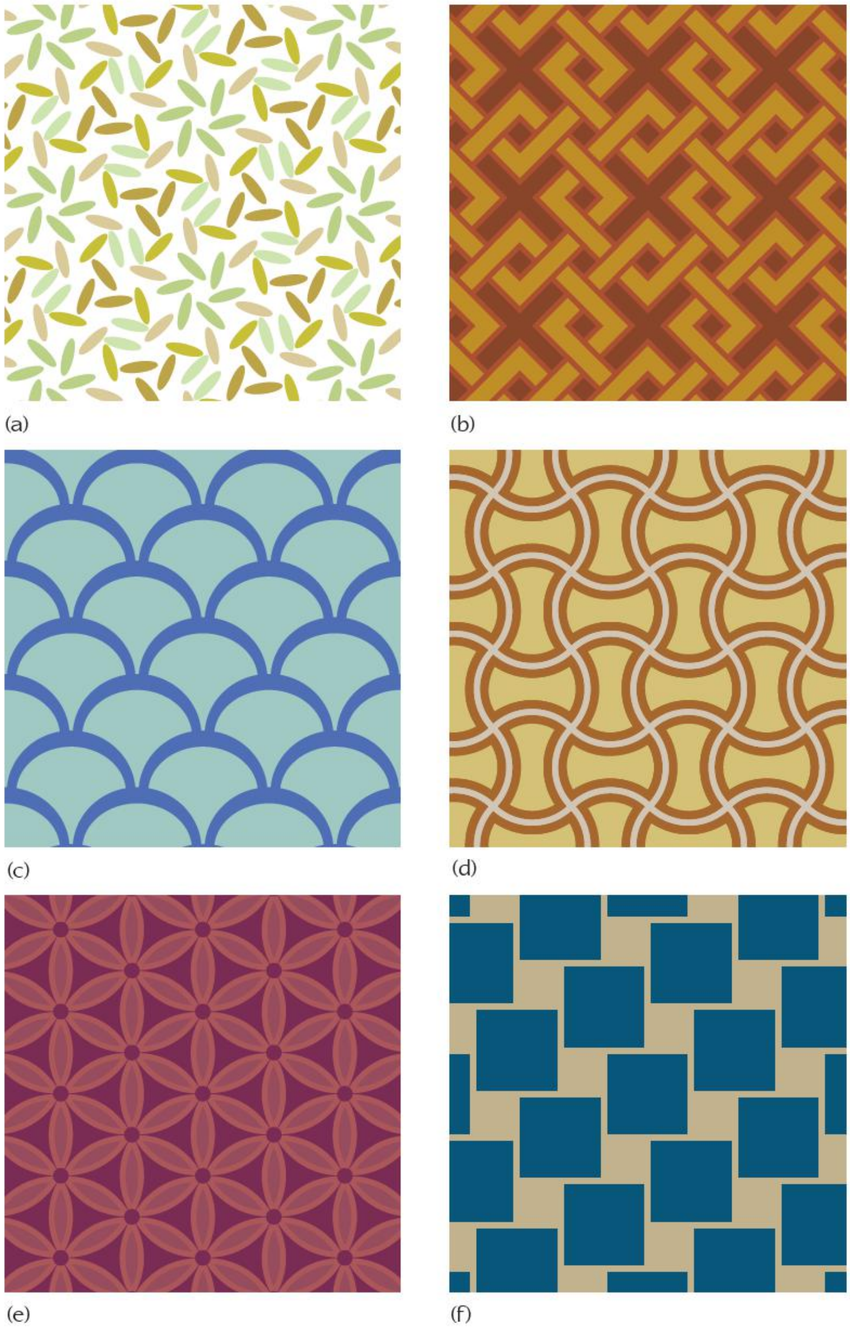
Problem 1.6. Find the signatures of these patterns. Mark any rotation points or mirror lines. $*42$ and $\mathbf{22}\times$



Problem 1.7. Try your hand at these: (a) $\mathbf{3} * 3$, (b) $\mathbf{22}\times$ (c) $\mathbf{4} * 2$ (d) $\mathbf{632}$ (e) $\mathbf{22}\times$, (f)$\mathbf{22}\times$

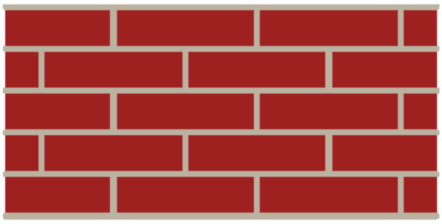


Problem 1.8. ...and these! (a) **632**, (b) **442**, (c) $*\times$, (d) **44** $*\,2$, (e) $*632$, (f) **442**

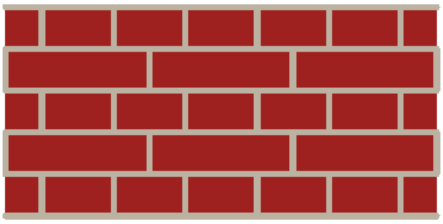


Exercise 1.9. On a separate paper, design a wallpaper pattern with signature $*442$.

Problem 1.10. Find the signatures of the brick patterns: (a) $\mathbf{2}*22$, (b) $*2222$ (c)$\mathbf{2}*22$ (d) $\mathbf{2}*22$ (e) $\mathbf{2222}$ (f) $\mathbf{22}*$ (g) $\mathbf{22}*$ (h) $\mathbf{22}*$



(a)



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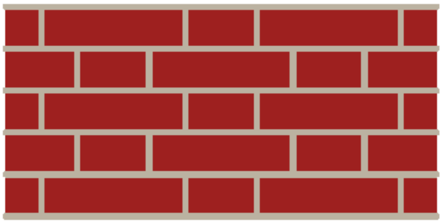

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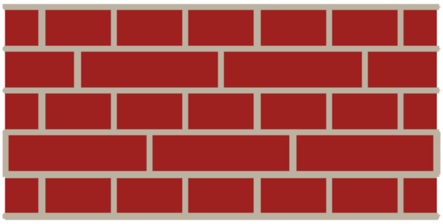

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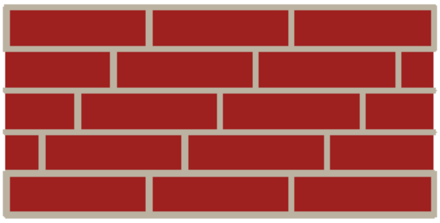

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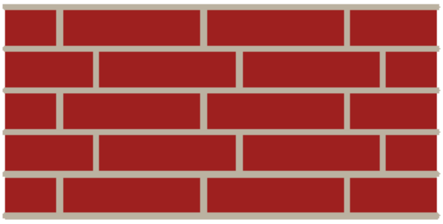

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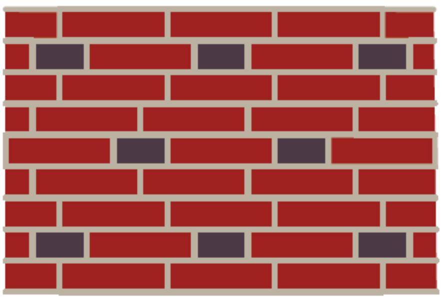

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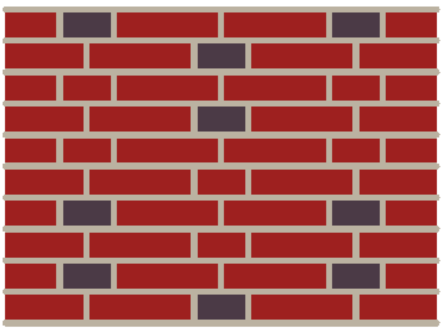


(h)

§2 The Classification Theorem

First, we associate a "cost" to each type of symmetry.

Symbol	Cost	Symbol	Cost
$\bigcirc$	2	$\times$ or $*$	1
2	$\frac{1}{2}$	2	$\frac{1}{4}$
3	$\frac{2}{3}$	3	$\frac{1}{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{n-1}{n}$	n	$\frac{n-1}{2n}$

Then we can calculate the total cost of a signature. For example, the pattern in Figure 1 with signature **333** has cost

$$\frac{2}{3}+\frac{2}{3}+\frac{2}{3}=2.$$

Problem 2.1. Calculate the costs of the signatures of Figures 2, 3, 4, and 6. Figure 2: cost of $\mathbf{3}*3$ is $\frac{2}{3}+1+\frac{1}{3}$. Figure 3: $1+1$. Figure 4: $1+1$, Figure 6: $\frac{3}{4}+1+\frac{1}{4}$

We will (eventually) prove this theorem:

Theorem 2.2 (Signature Cost Theorem)
The signatures of planar wallpaper patterns are exactly the ones with total cost 2.

For now, accept the theorem and you can classify *all* the planar signatures!

Problem 2.3. Among the 4 symmetries, which preserve orientations? Which type reverse orientations? Reflections and glide reflections reverse orientations (directions of spirals). Translation and rotation preserve orientations.

Problem 2.4. Use the signature-cost theorem to find all the signatures consisting of only $\bigcirc$ or rotational symmetries. **632**, **442**, **333**, **2222**, $\bigcirc$

Problem 2.5. Find all the signatures consisting of only mirror symmetries.

Problem 2.6. Find all the remaining signatures: all must be "hybrids" of mirror symmetries, rotational symmetries, or $\times$. (*Hint: They are all shown in Figure* 8.)

Problem 2.7. Find the signatures of these hybrid types. $*333, *442, *632, *2222, **$.

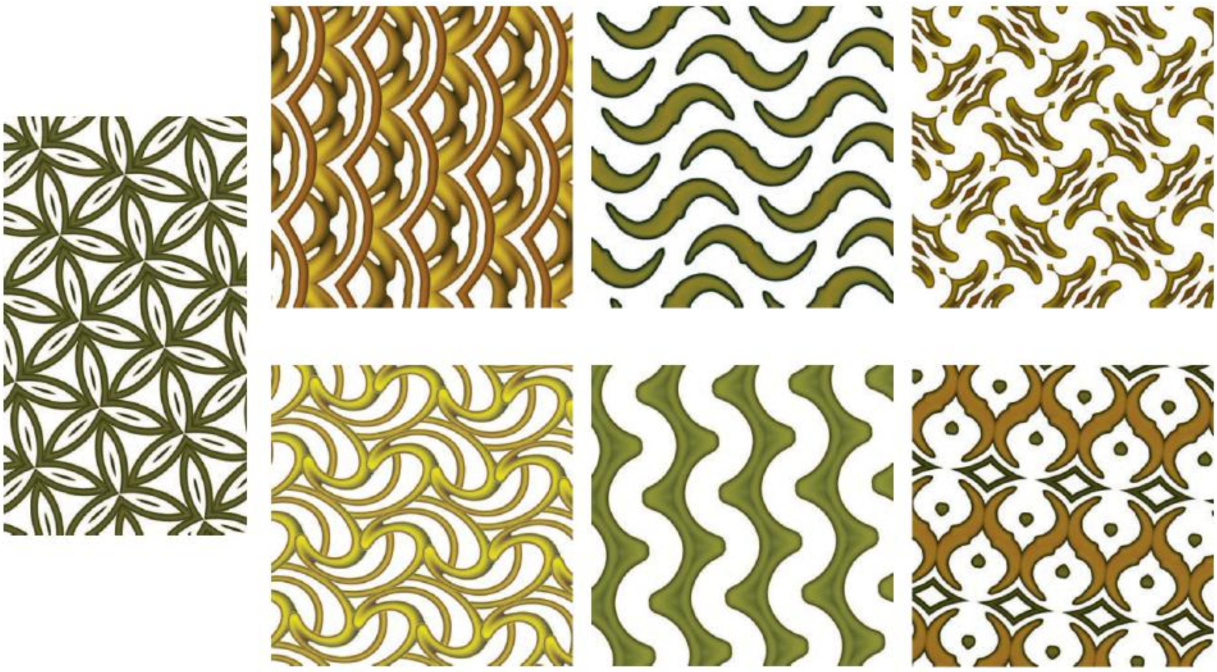
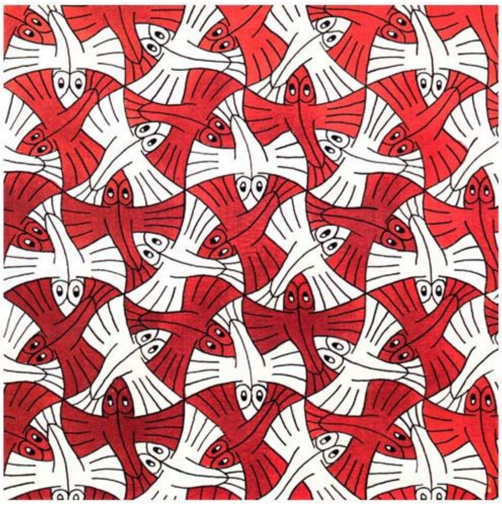


Figure 8: All the "hybrid" types

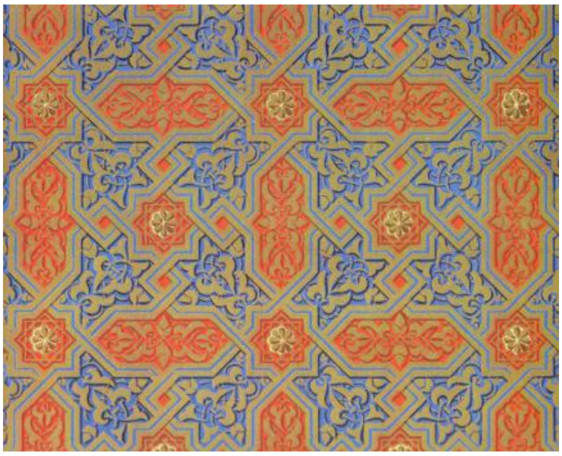
§3 Extra Problems

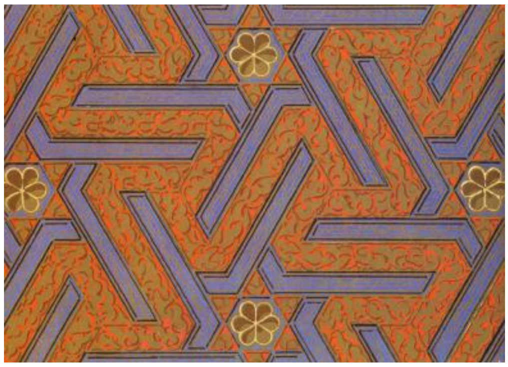
Find the signatures of these miscellaneous patterns. Use the Signature-Cost Theorem to help you

Problem 3.1. Some tilings by M.C. Escher: **333**, and **2222**

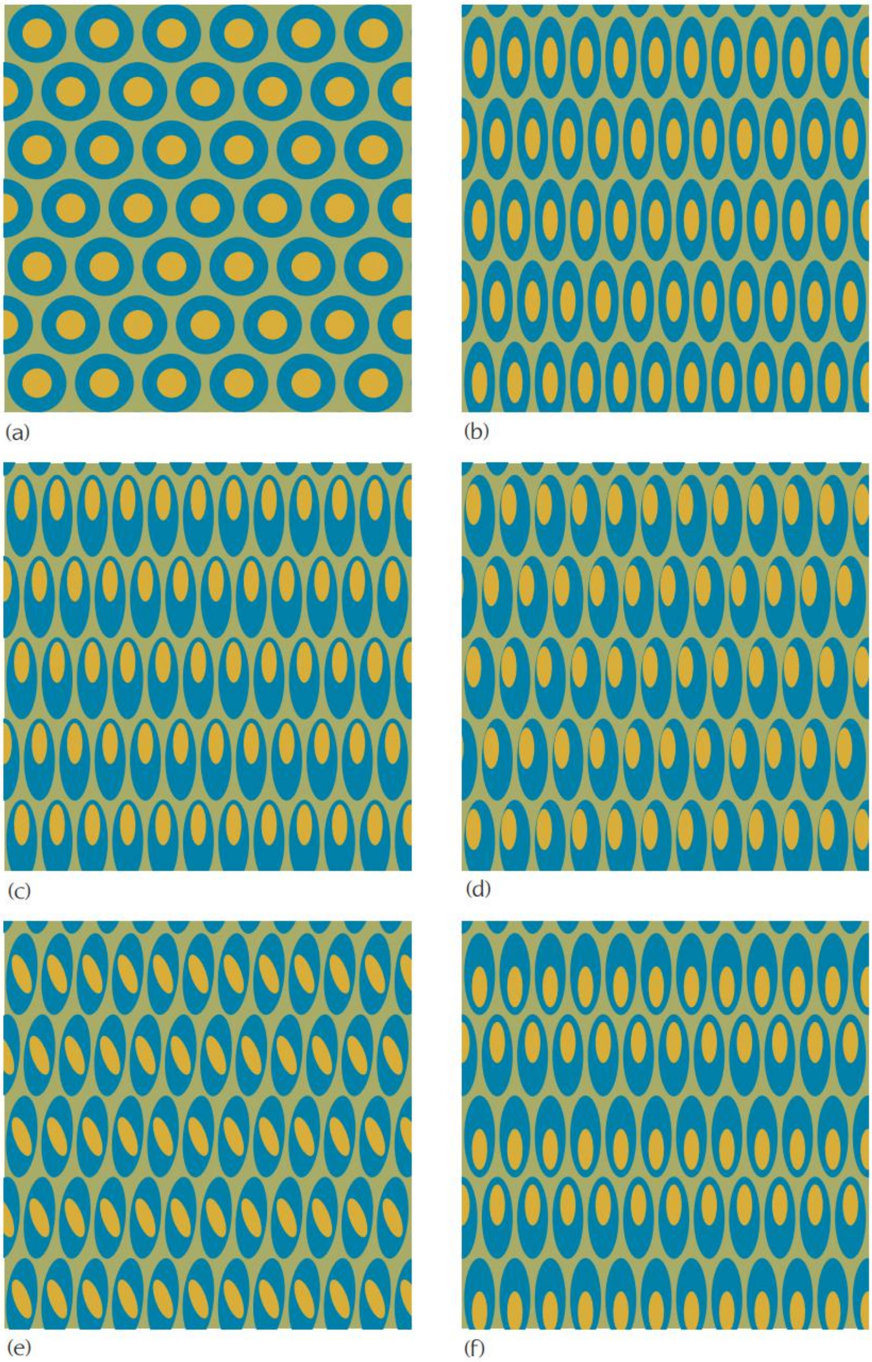


Problem 3.2. The Alhambra is an Islamic palace built in the 13th century. Amazingly, all of the wallpaper patterns are found in its design. **442**, and **632**





Problem 3.3. Some polka-dot patterns: (a) $*632$ (b) $\mathbf{2}*22$ (c) $*\times$ (d) $\bigcirc$ (e) $\mathbf{2222}$ (f) $\mathbf{22}*$



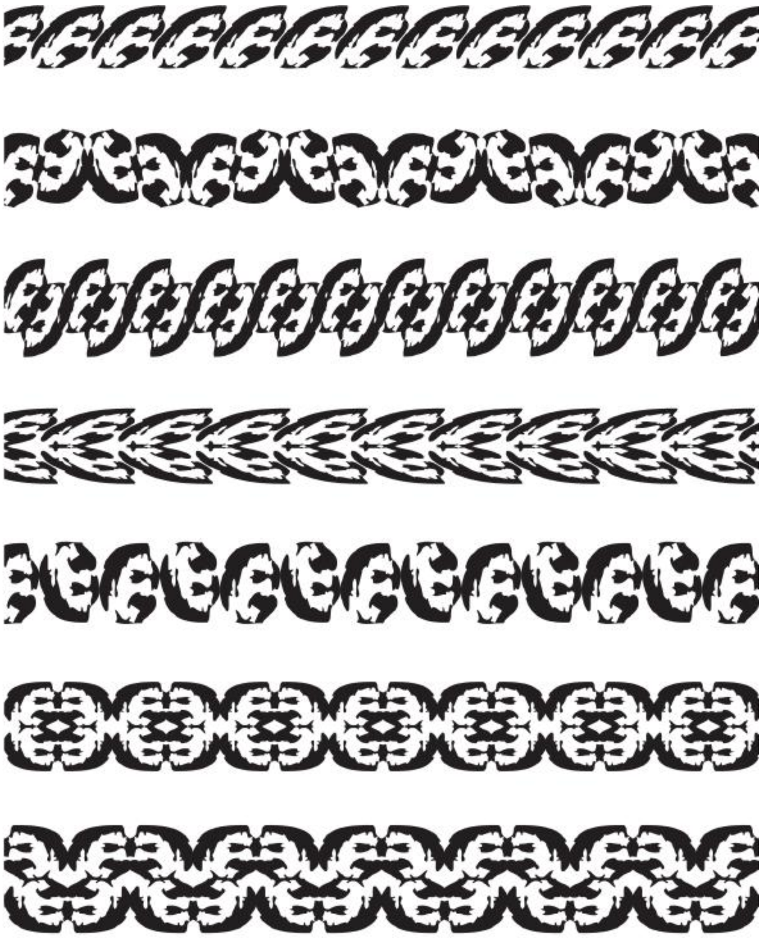
§4 Frieze Patterns

Frieze patterns are band patterns that have translational symmetry in only one direction rather than two. There are two new types of symmetries we label. Imagine wrapping the frieze band around the equator of a sphere. If the "north pole" or "south pole" is a reflection point, then we denote this as $*\infty$. If one of them is a rotation, we denote this as ∞. Remember that two reflection or rotation points are the same if we can move one onto the other without changing the pattern (in this case, by rotating the sphere).

Theorem 4.1 (Signature Cost for Friezes)

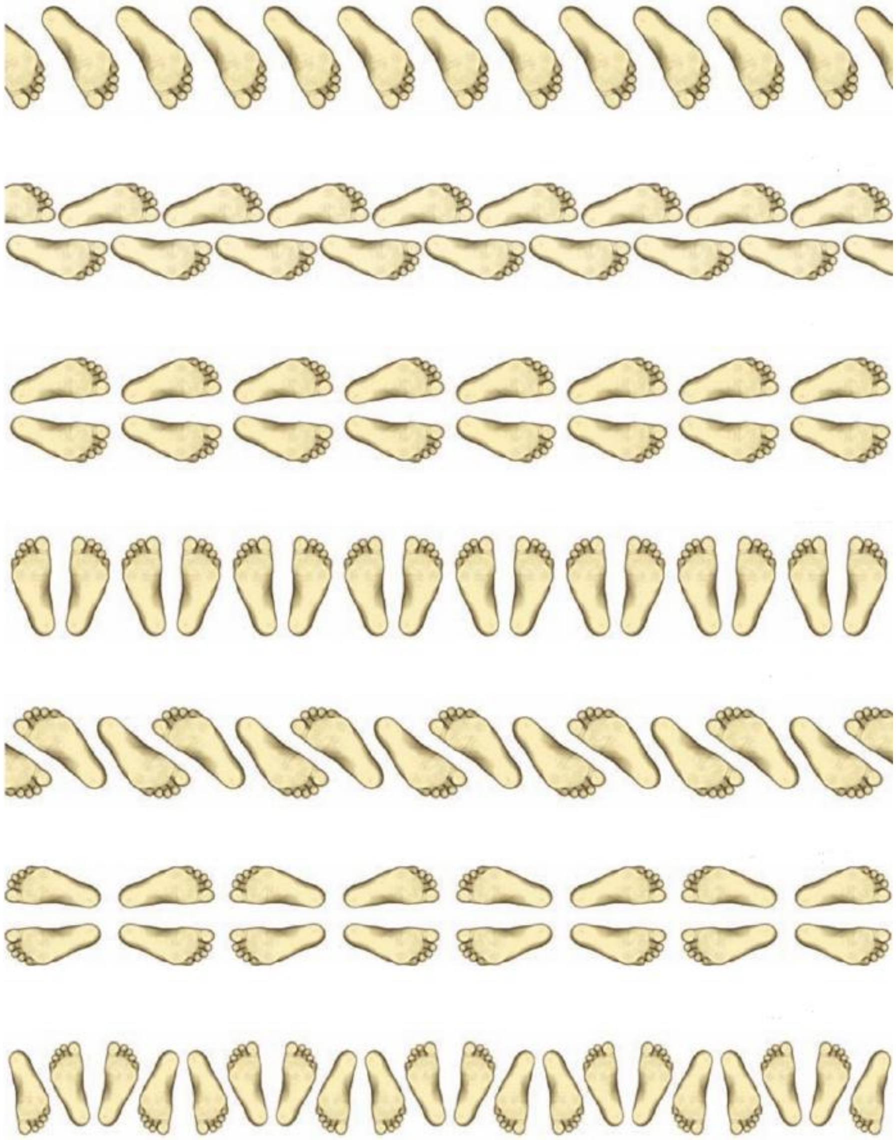
Set the cost of $*\infty$ to $\frac{1}{2}$ and the cost of ∞ to 1. Then the signatures of frieze patterns have total cost 2.

Problem 4.2. Find the signatures of these frieze patterns
$\infty\infty$, $\mathbf{2*\infty}$, $\mathbf{22\infty}$, $\infty*$, $\infty\times$, $*22\infty$, $\mathbf{2*\infty}$.



Problem 4.3. Find all the signatures of frieze patterns (with proof). All the signatures are in Problem 4.4

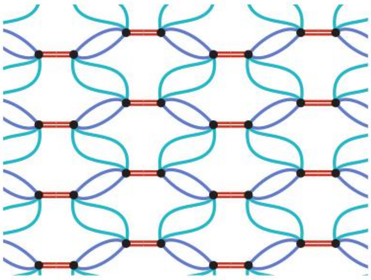
Problem 4.4. More practice with frieze patterns: $\infty\infty$, $\mathbf{2}*\infty$, $\mathbf{22}\infty$, $\infty*$, $\infty\times$, $*22\infty$, $\mathbf{2}*\infty$.



§5 Group Presentations

Problem 5.1. Write down a group presentation for the group G corresponding to the signature $*632$. Show that the group corresponding to **632** is a subgroup of G.
$G = \langle P, Q, R | \, 1 = P^2 = (PQ)^6 = Q^2 = (QR)^3 = R^2 = (RP)^2 \rangle$. The subgroup generated by QR, RP, and QP corresponds to the rotational **632** group.

Problem 5.2. Find signatures and come up with group presentations of the following patterns: 1) **22**$\times$. $1 = \alpha^2 = \beta^2 = P^2$, $P\alpha\beta = \alpha\beta P$. 2) $\bigcirc$. $XY = YX$ 3) **22** $\times$. $1 = (YZ)^2 = (YZ^{-1})^2$. 4) **22** $\times$. $1 = \alpha^2 = \beta^2 = \infty BZ^2$. 5) $**$. $1 = P^2 = Q^2, \alpha P = P\alpha$, $\alpha Q = Q\alpha$. 6) $\times\times, Y^2Z^2 = 1$



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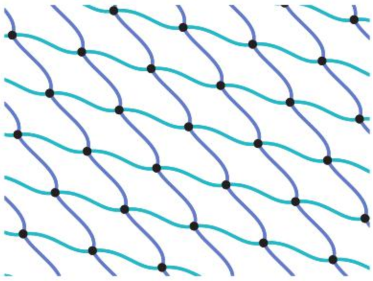

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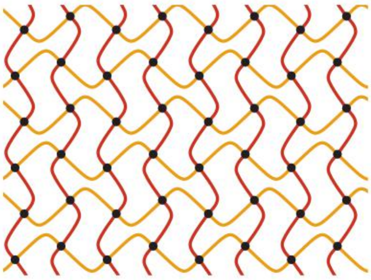

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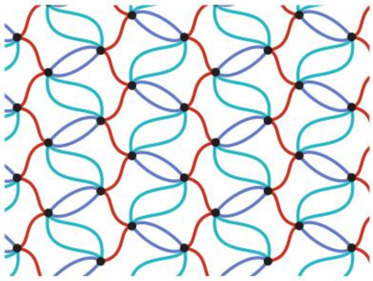

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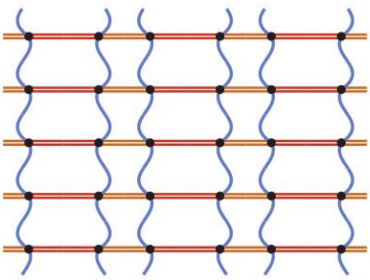

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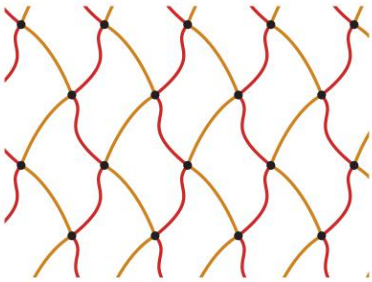

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