

Euler paths, Euler Circuits, and Dual Graphs

April 20, 2014

Warm Up Problem (Oldaque P. de Freitas Puzzle)

Two ladies are sitting in a street cafe, talking about their children. One lady says that she has three daughters. The product of the girls' ages equals 36 and the sum of their ages is the same as the number of the house across the street. The second lady replies that this information is not enough to figure out the age of each child. The first lady agrees and adds that her oldest daughter has beautiful blue eyes. Then the second lady solves the puzzle. Please do the same.

1. The goal of this problem is to prove the following property of graphs

The number of odd vertices in any graph is even.

First, note that to get any graph with n vertices we can do the following

- start with n vertices (and no edges);
- add edges one at a time, noting how the number of odd vertices changes with each new edge;

(a) How many odd vertices are there in a graph with n vertices and no edges?

(b) Suppose we start adding edges one at a time.

i. How does the number of odd vertices change when we add an edge between two even vertices?

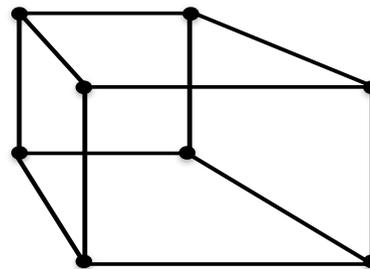
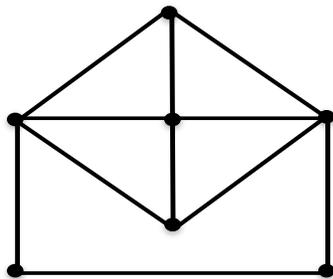
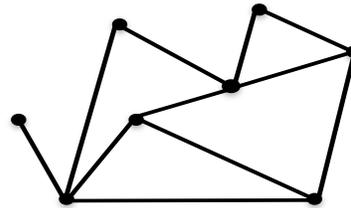
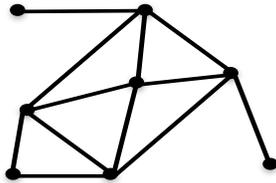
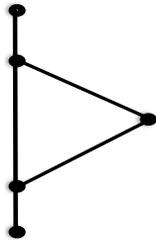
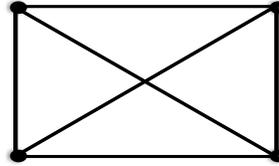
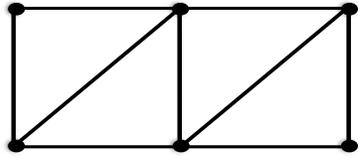
ii. How does the number of odd vertices change when we add an edge between two odd vertices?

iii. How does the number of odd vertices change when we add an edge between an even and an odd vertex?

(c) Based on your answers from parts (a) - (b), what must be true about the number of odd vertices in any graph?

Euler paths

2. Circle the graphs that have Euler paths. Draw Euler paths on the graphs (indicating the starting and the ending point).



Label the degrees of all the vertices.

What is true about the degrees of the vertices for Euler paths that are not Euler Circuits?

3. Emmanuelle stated that

Graphs which have Euler paths that are not Euler Circuits must have two odd vertices.

Let's figure out if she is correct. We can think of the edges at a vertex as "entries" and "exits". In other words, edges can be used to "enter" or "exit" a vertex.

For a graph that has an Euler path, we have three type of vertices: starting vertex, intermediate vertices, and end vertex.

(a) What can you say about the number of entries and exits at each intermediate vertex? Explain your answer.

(b) Are intermediate vertices even or odd?

(c) What can you say about the number of entries and exits at the starting vertex? Is the starting vertex even or odd?

(d) What can you say about the number of entries and exits at the ending vertex? Is the ending vertex even or odd?

(e) How many odd vertices can there be in Euler paths that are not Euler Circuits?

4. Below are tables that list the degrees of vertices for different graphs. For each table, write down whether an Euler path exists, and explain why or why not.

(a)

Vertex	Degree
1	2
2	2
3	4
4	3
5	3
6	4

(b)

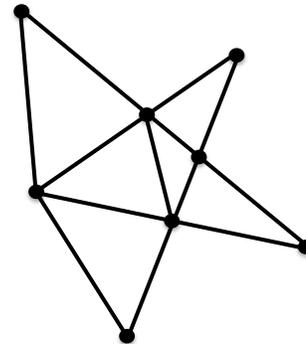
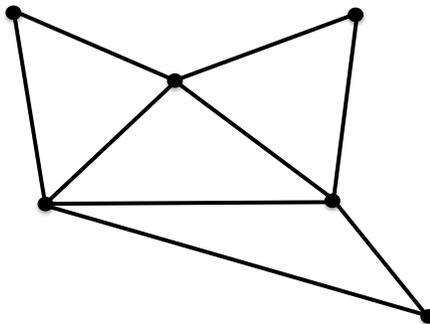
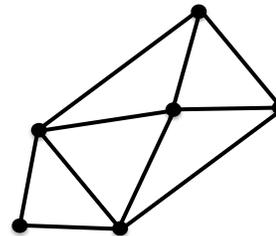
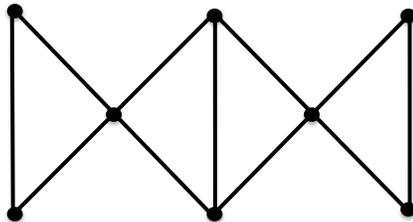
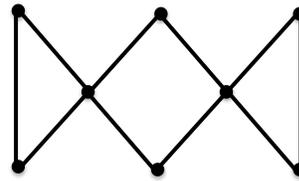
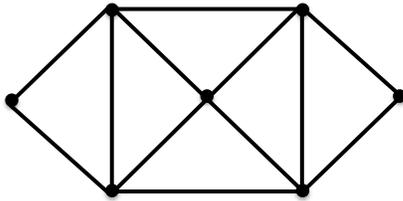
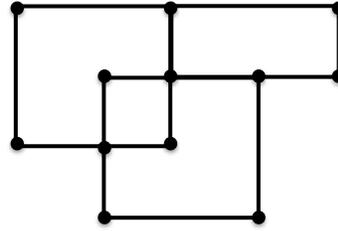
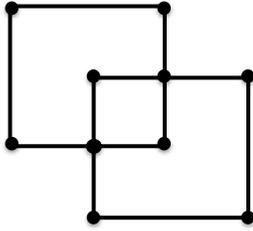
Vertex	Degree
1	3
2	3
3	4
4	3
5	2
6	3
7	2

(c)

Vertex	Degree
1	2
2	4
3	3
4	2
5	4

Euler Circuits

5. Circle the graphs below that have Euler Circuits. Draw Euler paths on the graphs (indicating the starting and the ending point).



6. What is true about the degrees of the vertices in graphs that have Euler Circuits? Explain using the “entries” and “exits” analogy.

7. Below are tables that list the degrees of vertices for different graphs. For each table, write down whether an Euler Circuit exists, and explain why or why not.

(a)

Vertex	Degree
1	2
2	2
3	3
4	3

(b)

Vertex	Degree
1	2
2	4
3	4
4	2
5	2

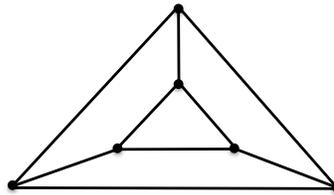
Dual Graphs

Given a graph, we can construct a new graph called the *dual graph*. A graph divides the plane into several regions. The dual graph is constructed in the following way:

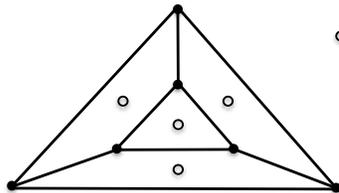
- Vertices of the dual graph correspond to regions into which the original graph divides the plane;
- Two vertices are connected by an edge if the corresponding regions are adjacent (i.e., share an edge in the original graph);

Here is an example of constructing the dual graph:

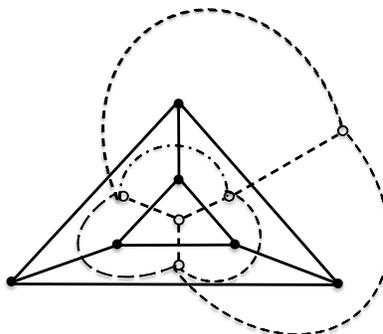
(i) Suppose the following graph is given:



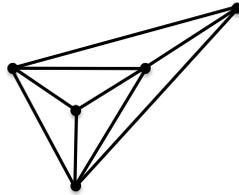
(ii) A vertex is placed in each region of the original diagram. Do not forget about the region outside of the graph!



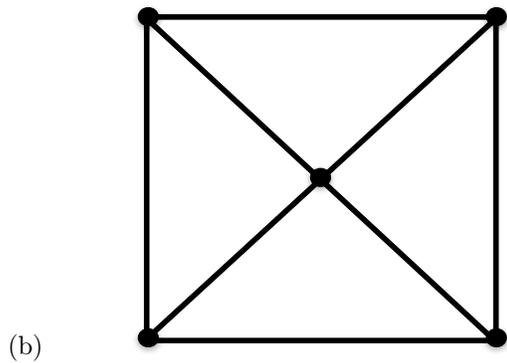
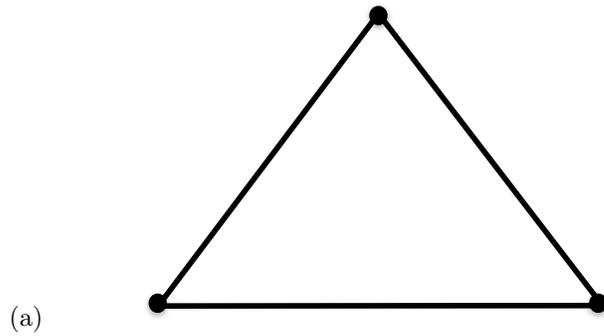
(iii) New vertices that are in adjacent regions of the diagram are connected with edges. (When you do this, each edge of the original graph has one edge of the dual graph crossing it).



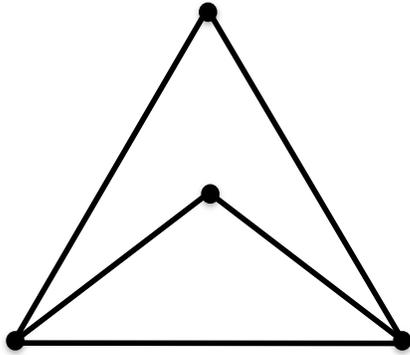
(iv) The new graph made is called the dual of the original graph. Straighten out the edges so the dual graph looks as follows:



8. Create the dual graphs of the following graphs by drawing the new vertices and edges on the graphs. Then, draw the dual graphs on the right of each corresponding graph.



9. (a) Create the dual graph of the following graph.



(b) Draw the dual of the dual graph.

(c) What do you notice about the original graph and the dual of the dual graph?

Extra Problem

10. Solve the following cryptarithms. Remember that different letters correspond to different digits, and same letters correspond to same digits. The first digit in each number cannot be zero.

$$\begin{array}{r} \text{T H I S} \\ + \quad \text{I S} \\ \hline \text{E A S Y} \end{array}$$

$$\begin{array}{r} \text{B A L L} \\ + \text{B A L L} \\ \hline \text{C A T C H} \end{array}$$