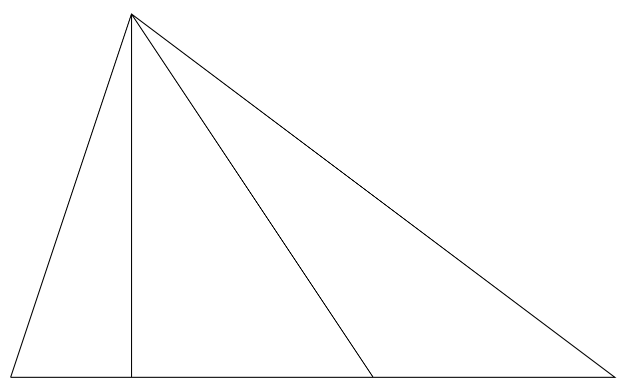
14 Odd and even numbers

Materials for the lesson and homework: a couple of regular pencils, an eraser, a pencil sharpener.

We study odd and even numbers in chapters 8 and 9 of [1], an LAMC enrichment textbook and workbook covering the first half of a school year for kindergarten and first grade. This chapter is totally independent from [1] although they share a few similar problems and proofs. In [1], our approach is based on the concept of a partition, a way to break a number into parts. In this book, we introduce odd and even numbers in a different way and we study them in both decimal and binary forms. If a student has studied odd and even numbers using [1] in the past, this chapter will give her a new perspective and deepen her understanding of the concept. Since this book is for an older age, most of the problems in this book are more challenging.

Warm-up

Problem 14.1 *How many triangles are there on the picture below?*



There are ______ triangles.

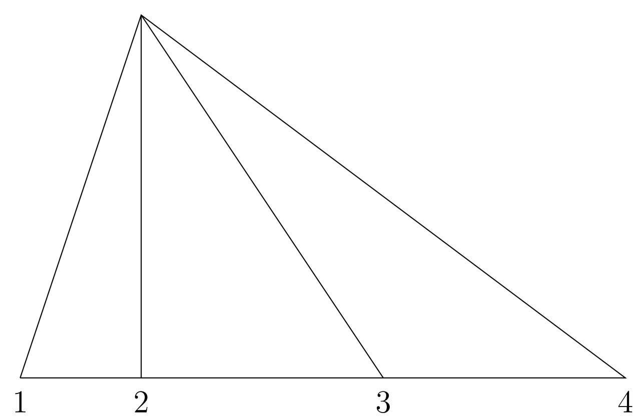
You will find two different solutions to problem 14.1 below. The first is more elementary, but harder to generalize. The second introduces students to two important combinatorial concepts, a *choice tree* and *double count*. A minor modification of the second solution works for problem 14.20 as well. Students are usually quick to discover the first, more elementary, solution on their own. This sets the stage for discussing the second solution. If your students are not yet ready for a higher level of abstraction, the second solution can be skipped.

Solution 1: let us call a triangle that has no smaller triangles inside a *triangle of size one*. Let us call a triangle that has two triangles of size one inside a *triangle of size two.* Let us call a triangle that has three triangles of size one inside a *triangle of size three*. There are three triangles of size one, two triangles of size two, and one triangle of size three on the picture above.

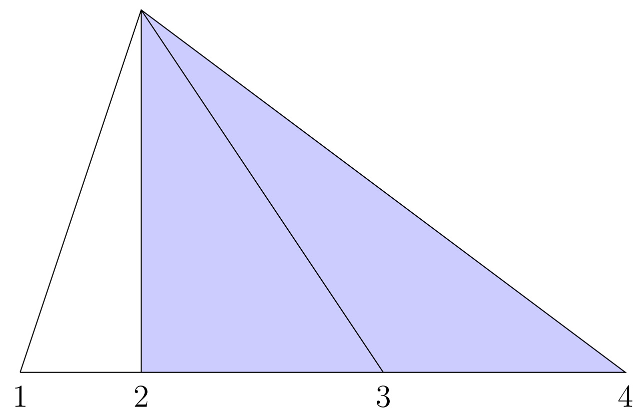
$$3 + 2 + 1 = 6$$

There are 6 triangles on the picture.

Solution 2: let us number the lines going from the upper vertex of the triangle down to the base.



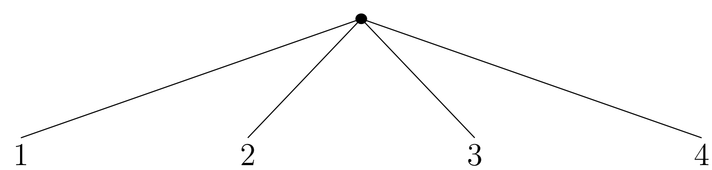
Note that to choose a triangle is the same as to choose a pair of its side walls. For example, choosing the lines 2 and 4 is equivalent to choosing the triangle shaded on the picture below.



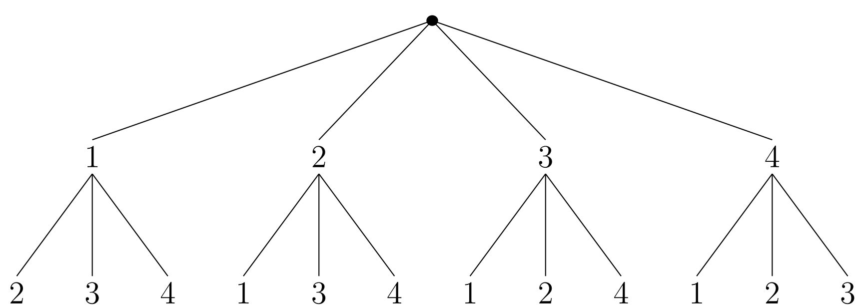
At this moment of time, please ask your students to show the triangles given by choosing the lines 1 and 2, 1 and 3, 2 and 3, 3 and 4, and 1 and 4. Observe that choosing the lines 2 and 4 is the same as choosing the lines 4 and 2.

The picture we about to construct is called a *choice tree*. It helps to solve many counting problems.

There are four ways to choose the first side of a triangle.



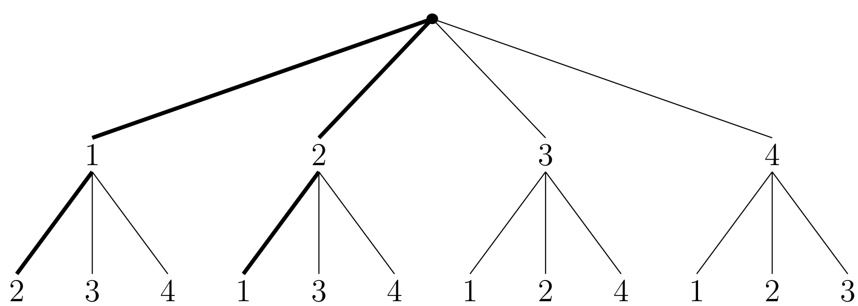
Once we choose the first side, we have three more choices for the second side. For example, if we choose side 1 first, we can choose either of the sides 2, 3, or 4 to be the second side of the triangle.



The top vertex of the tree is called its *root*. In math, trees have roots at the top, not at the bottom!

The number of choices equals the number of ways to travel from the root of the tree down. If your students are not familiar with multiplication, they can count the number of the bottom nodes. If they are familiar with multiplication, please point out that there are three choices of the second side for each of the four choices of the first side of the triangle. Therefore, the total number of the choices is $4 \times 3 = 12$.

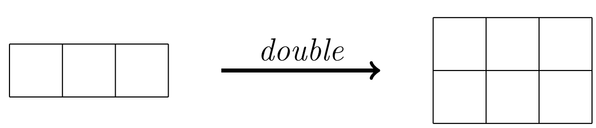
Note that we have counted each of the triangles twice. For example, choosing first side 1 then side 2 gives you the same triangle you get by choosing first side 2 then side 1.



This way, the number of all the triangles on the original picture equals a half of 12, that is 6.

Lesson

Recall that to *double* a number is to add the number to itself. For example, the double of 3 is $6 = 3 + 3$.



Doubling the number 64 and finding a number such that its double equals 64 in problem 14.2 should not be hard for students. 64 is a basic binary number they have played with in the past. If doubling the number 2019 and finding the number such that its double equals 250 is yet something your students cannot do with ease, please replace them by smaller numbers.

Problem 14.2

- *What is the double of the number* 64*?*

  The double is the number ______ .

- *What is the double of the number* 2019*? If needed, use the abacus to figure out.*

  The double is the number ________ .

- *Alice doubled a number and got* 64*. What was the original number?*

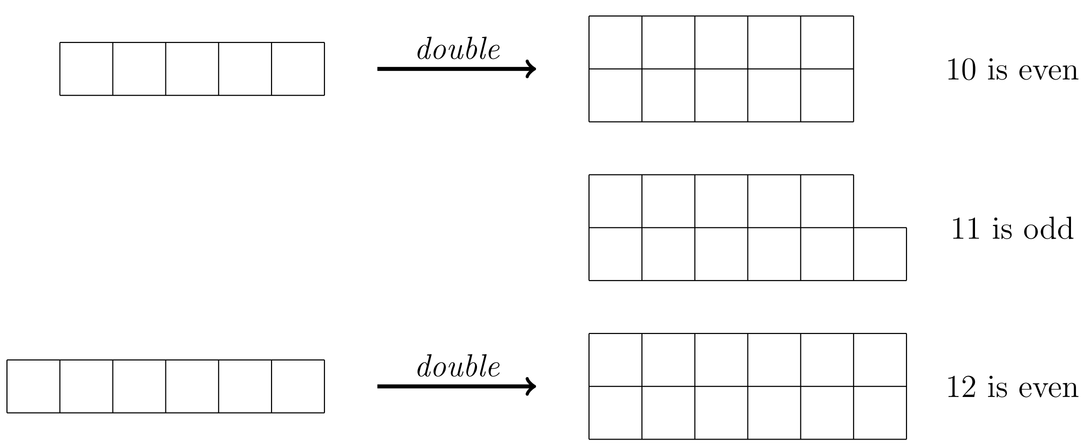
  The original number was ______ .

- *Cindy doubled a number and got* 250*. What was the original number?*

  The original number was ______ .

An integer is called *even*, if it is a double of another integer. An integer is called *odd* otherwise. For example, the number 10 is even because $10 = 5 + 5$. The number 11 is odd for the following reason: $5+5 = 10$ and $6+6 = 12$. Note that 11 is between 10 and 12, but there is no integer between 5 and 6. Thus, there is no integer one can double to get 11. So, 11 is not even. Therefore, it is odd.

The following picture helps to visualize the above explanation. It shows that the numbers 10 and 12 are doubles of other integers, but the number 11 is not.



Problem 14.3 *Is the number 0 odd or even? Circle the correct answer.*

Odd *Even*

Explain your choice.

Question 14.1 *What is a natural number?*

Problem 14.4

- *Is the number 22 odd or even? Circle the correct answer.*

Odd *Even*

Explain your choice.

- *Is the number 21 odd or even? Circle the correct answer.*

Odd *Even*

Explain your choice.

Problem 14.5

- *Write down the first five basic binary numbers in decimal notations.*

________ , ________ , ________ , ________ , ________

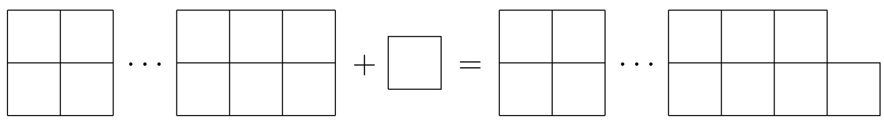
- *Circle odd basic binary numbers. Do not circle the even ones. Explain your choice.*

Solution: every basic binary number, except for 1, is a double of the previous binary number by construction. 1 is odd, the rest are even.

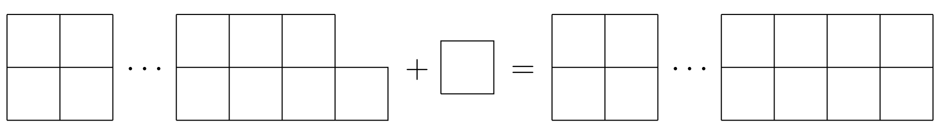
Problem 14.6 *Use pictures similar to the one on page 5 to explain the following formulas.*

- **even + 1 = odd**
- **odd + 1 = even**

Solution: a positive even number is a double of another positive integer. Therefore, it can be represented pictorially as a two-story house with an equal number of apartments on both floors. The dots in between the parts of the houses below show that we do not know the number of apartments in the houses, we only know the houses' shapes. Adding one more apartment to an "even" house results in a two-story house that cannot have the same number of apartments on both floors. Therefore, the number of the apartments in this house it's not a double of another integer and cannot be even. Thus, it's odd.

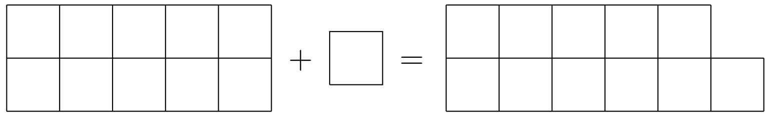


To prove the second formula, let us observe that an odd natural number cannot be represented by a two-story house with an equal number of apartments on each floor. The best we can do is to have one extra apartment on the first floor. Adding one more apartment to such a house provides the extra first floor apartment with a missing partner. The resulting two-story house has the same number of apartments on each floor. Therefore, the number of the apartments in the house is a double of another natural number. Thus, the result is even.



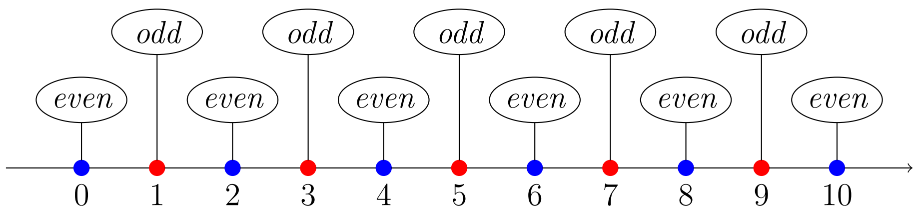
Initially, students will try to prove the problem 14.6 formulas using examples, like $2+1=3$ and $3+1=4$. Please point out to them that an example is not a proof. It seems that the problem 14.6 formulas are correct for some small numbers. However, there is no guarantee that for some really big numbers, so big that we cannot even write them down, the formulas still continue to work. We cannot check by inspection, so we need a proof.

Once students start learning the pictorial argument presented in this book, they may come up with the following "proof".



Please let them know that this argument is in fact an example, like $10 + 1 = 11$ on the picture above. It's the dots in between the parts of the houses than show that we need not to know the number of apartments in a house to carry out the argument. All we need is to know the house's shape.

Problem 14.6 shows that odd and even numbers alternate on the number line.



Problem 14.7 *Explain the following formulas.*

- **even + even = even**
- **odd + even = odd**
- **odd + odd = even**

A pictorial argument similar to the one used to solve problem 14.6 works for this problem as well.

Two numbers are called *consecutive*, if one of them follows the other as we count. For example, the numbers 9 and 10 are consecutive.

Question 14.2 *Why did we study optical illusions in a math class?*

Problem 14.8 *Alice wrote two consecutive numbers on the board. One of the numbers is 25. What is the other number?*

Solution: either 24 or 26.

Problem 14.9 *The sum of two consecutive numbers equals* 15. *What are the numbers?*

The numbers are ______ and ______ .

Problem 14.10 *Bob has a book open in front of him. The sum of the numbers of the pages he is looking at is* 29. *What are the numbers of the pages Bob is looking at?*

The numbers are ______ and ______ .

Problem 14.11 *Is the sum of two consecutive numbers always odd, always even, or can it be either odd or even? Circle the correct answer.*

Odd *Even* *Either*

Explain your choice.

Solution: we have shown in problem 14.6 that odd and even numbers alternate on the number line. Please also see the number line picture on page 8. If the first of the consecutive numbers is odd, then the second is even. As shown in problem 14.7, **odd + even = odd**. If the first of the consecutive numbers is even, then the second is odd. Once again, **odd + even = odd**.

Problem 14.12 *Binary Bob has a magazine open in front of him. The sum of the binary numbers of the pages he is looking at is B* 101. *What are the binary numbers of the pages Binary Bob is looking at?*

The numbers are B ______ and B ______ .

Solution: The number B 101 is a sum of the basic binary numbers 4 and 1, B 101 = 5. The latter number is a sum of two consecutive number, $5 = 2 + 3$. Since 2 = B 10 and 3 = B 11, Binary Bob is looking at the pages numbered B 10 and B 11.

Problem 14.13 *For the binary numbers below, decide whether each of them is odd or even, circle the correct answer and explain your choice.*

- $B\ 10:$ $\qquad Odd \qquad Even$

- $B\ 101:$ $\qquad Odd \qquad Even$

- $B\ 110:$ $\qquad Odd \qquad Even$

- $B\ 1011:$ $\qquad Odd \qquad Even$

This problem can be solved in two different ways. The first is to convert a binary number to the decimal form and then to decide whether it is odd or even. For example, B 1011 $= 1 + 2 + 8 = 11$. We have shown that 11 is an odd number at the beginning of this lesson.

Another way is to recall problem 14.5. Solving the problem, we noticed that while 1 is odd, all other basic binary numbers are even. Since **even + even = even**, a binary number is odd if and only if 1 is a part of its decomposition. 1 is a part of a number's binary decomposition if and only if the last digit of the number in the binary form is 1. If this is not the case, the last digit is 0. So, binary numbers ending with 0 are even. Binary numbers ending with 1 are odd. This consideration solves problem 14.14 as well.

Problem 14.14

- *A binary number ends with* 1. *Is it odd, even, or can it be either? Circle the correct answer.*

$Odd \qquad Even \qquad Either$

Explain your choice.

- *A binary number ends with* 0. *Is it odd, even, or can it be either? Circle the correct answer.*

Odd *Even* *Either*

Explain your choice.

A rule similar to the one derived in problem 14.14 is not hard to derive for decimal numbers.

Problem 14.15

- *Write down the first five basic decimal numbers.*

______ , ______ , ______ , ______ , ______

- *Circle odd basic decimal numbers. Do not circle the even ones. Explain your choice.*

Problem 14.15 shows that all the basic decimal numbers except for 1 are even. 10 is a double of 5, 100 is a double of 50, 1000 is a double of 500, and so on.

Example 14.1 *To figure out whether the number 2019 is odd or even, let us use its decimal decomposition. We have figured it out in problem* **??**.

studied number	*basic decimal numbers*				
	10000	**1000**	**100**	**10**	**1**
2019		2	0	1	9

2019 = *two 1000s + one 10 + nine 1s. We know that all the basic decimal numbers except for 1 are even. Therefore, the sum of any number of tens, hundreds, thousands, etc. is even. The last digit of the number 2019 is odd. We know that* **odd + even = odd**. *Therefore, 2019 is an odd number.*

Problem 14.16

- *Write down all the decimal digits starting from zero.*

________ , ________ , ________ , ________ , ________

________ , ________ , ________ , ________ , ________

- *Circle the odd digits. Do not circle the even ones. Explain your choice.*

We can now formulate the rule that tells you whether an integer is odd or even in both the binary and decimal systems.

An integer is even if and only if its last digit is even. An integer is odd if and only if its last digit is odd.

In the binary, an integer is even if and only if its last digit is 0. An integer is odd if and only if its last digit is 1.

In the decimal system, an integer is even if and only if its last digit is one of the following: 0, 2, 4, 6, 8. An integer is odd if and only if its last digit is one of the following: 1, 3, 5, 7, 9.

Problem 14.17 *The string below is a mixture of decimal and binary numbers. Circle even numbers. Do not circle odd numbers.*

$$273, \quad 1024, \quad B\,110111, \quad 698, \quad B\,101101110$$

Problem 14.18 *Snow White wrote the number* 20 *on a piece of paper and gave it to the seven dwarfs. Each of the dwarfs either added or subtracted 1 from the number he got, marked out the old number and wrote down the new one.*

- *Could they get* 21 *as the final number? If you think they could, show how. If you think they couldn't, explain why.*

- *How about* 23*?*

- *How about* 22*?*

- *How about* 18*?*

- *How about* 17*?*

- *List all the numbers the dwarfs could get.*

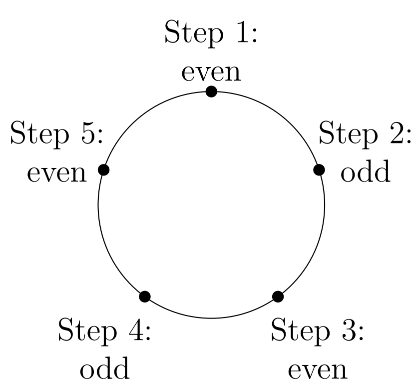
Solution: Let us suppose for starters that all the seven dwarfs subtract 1. In this case, the final result is 13. If six dwarfs subtract 1 while one dwarf adds 1, the result is two more than 13. Indeed, the dwarf does not subtract 1, but adds it instead. This increases 13 by 2 and produces 15 as the final result. If two dwarfs add 1, the last number increases by 2 for the same reason, and so forth. Therefore, the possible answers are 13, 15, 17, 19, 21, 23, 25, and 27.

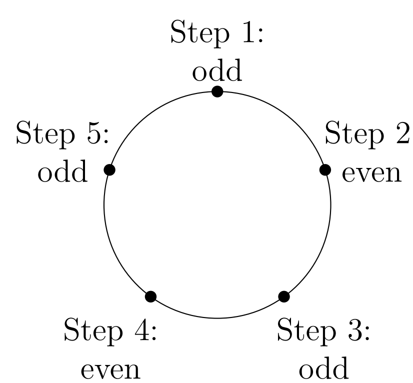
Problem 14.19

- *There are five natural numbers on a circle. Show that you can always find a pair of neighboring numbers with an even sum.*

- *There are 2019 natural numbers on a circle. Show that you can always find a pair of neighboring numbers with an even sum.*

Solution: to solve the first part of the problem, let us try to find five natural numbers on a circle such that there are no two neighbors with an even sum. Let dots represent the numbers.





Step 1: suppose the number represented by the top dot is even. Please see the left-hand side picture above. If the next number in the clock-wise direction from this one is also even, we have a pair with an even sum. To avoid this situation, the neighboring number must be odd.

Step 2: the next number in the clock-wise direction is odd. If the next number in the clock-wise direction from this one is also odd, then their sum is even. To avoid this situation, the neighboring number must be even.

Step 3: the next number in the clock-wise direction is even. Following the same logic, the next number in the clock-wise direction from this one must be odd and the number further in the clock-wise direction from the latter one must be even.

Step 4: odd.

Step 5: even.

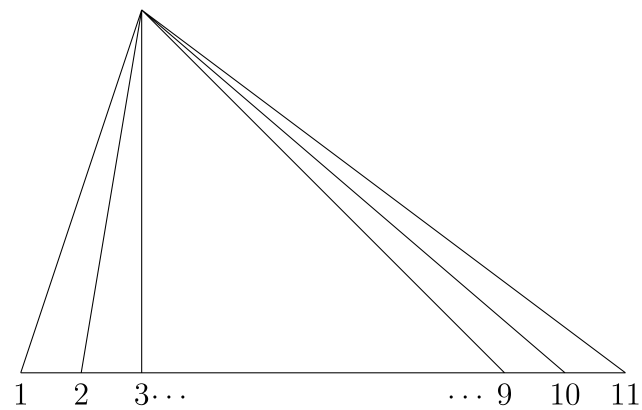
Now the top number and the number to the left of it are both even. They have an even sum!

Suppose the number represented by the top dot is odd. Please see the right-hand side picture above. Acting as in the previous case, we end up having two odd neighboring numbers. Their sum is even. There is no way to avoid it!

To solve the second part of the problem, let us take a second look at the method we used to solve the first part. The method can only work if the numbers come in either odd-even or even-odd pairs. This means that for the method to work, the total number of the dots on the circle must be even. We have seen in example 14.1 that the number 2019 is odd. Thus, there is no way to avoid having a pair of neighboring numbers with an even sum.

Give the following problem to a student only if he knows multiplication.

Problem 14.20 ❯ *Lines from the top vertex to the base of a triangle are numbered from* 1 *to* 11 *as shown on the picture below. How many triangles are there on the picture?*



Solution: $11 \times 10 \div 2 = 55$. See the second solution to problem 14.1.

Homework

Finish all the problems from class.

Problem 14.21

- *Is the number* 102 *odd or even? Circle the correct answer.*

Odd *Even*

Explain your choice.

- *Is the number* 103 *odd or even? Circle the correct answer.*

Odd *Even*

Explain your choice.

Question 14.3 *What numbers do we call consecutive?*

Problem 14.22

- *The sum of two consecutive numbers equals* 43*. What are the numbers?*

The numbers are ______ *and* ______ *.*

- *The sum of three consecutive numbers equals* 63*. What are the numbers?*

The numbers are _____ *,* _____ *, and* _____
.

Problem 14.23 *Binary Beth has a book open in front of her. The sum of the binary numbers of the pages she is looking at is B* 1101*. What are the binary numbers of the pages Binary Beth is looking at?*

The numbers are B ______ *and B* ______
.

Problem 14.24 *For the binary numbers below, decide whether each of them is odd or even without converting to decimals. Circle the correct answer and explain your choice to your parents.*

- *B* 111010 : *Odd* *Even*
- *B* 1011001 : *Odd* *Even*
- *B* 10000010 : *Odd* *Even*
- *B* 11111011 : *Odd* *Even*

Problem 14.25 *Is the sum of the numbers*

$$1 + 2 + 3 + \ldots + 98 + 99 + 100$$

odd or even? Circle the correct answer.

$$Odd \qquad\qquad Even$$

Explain your choice.

Solution: there are 50 odd and 50 even numbers in the sum. The sum of any number of even numbers is even. Let us group 50 odd numbers into 25 pairs. The sum of two odd numbers in each pair is even. Therefore, the final sum is even.

Problem 14.26 *There are five numbers such that the sum of any three of them is even. Show that all the numbers are even.*

Solution: we show this by contradiction. We assume that at least one of the five numbers is odd and show that this leads to a contradiction with the requirement of the problem for the sum of any three numbers out of five to have an even sum.

Assume one of the five numbers is odd. Let us call the number a. If a is the only odd number then the sum of a and any other two numbers is odd because the sum of the latter two numbers is even and **odd + even = odd**. Therefore, there should be one more odd number among the five. Let us call it b. The number $a + b$ is even because **odd + odd = even**. Let us call the remaining three numbers c, d, and e. Since the numbers $a + b + c$, $a + b + d$, and $a + b + e$ must be even, the numbers c, d, and e must be even. If this is the case however, the number $a + c + d$ is odd, but it must be even. The contradiction shows that the original assumption is incompatible with the requirement of the problem.

Problem 14.27 *A teacher has organized a library in her classroom. Currently, there are 40 books on the library's shelves. There are 23 students using the library. Within a week, every student either checks out a book or returns one to the library. Could there be 24 books on the library shelves at the end of the week? Circle the correct answer.*

$$Yes \qquad\qquad No$$

Explain your choice.

Solution: no, there couldn't. This problem is very similar to problem 14.18.

References

[1] O. Gleizer, O. Radko, *Breaking Numbers into Parts*, Second Edition, Part 1

[2] O. Gleizer, O. Radko, *Breaking Numbers into Parts*, Second Edition, Part 2