Braids, Knots and String Things

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1 Knots

Last week, we defined a knot as a closed piece of string, and represented them with diagrams, which were their projections onto the plane. In these diagrams, we drew arcs between points and noted which string passed over/under at each crossing. Let's warm up with some simple problems about knots.

Problem 1.1. Could you convey the information about a trefoil to someone else sitting at your table without drawing?

Problem 1.2. Draw a knot. Can you color the regions inside the knot black and white such that no two adjacent regions have the same coloring (i.e, a checkerboard coloring)? Can you do this for all knots?

Problem 1.3. What is the smallest number of crossings a knot can have and be non-trivial?

2 Colorings

We called two knots "equivalent" if we could find a smooth deformation from one knot to another. We used 3 moves to deform the knots and called them Reidemeister Moves (see Figure 1); they were the only moves that were required to get from any one knot diagram to another.

We began looking at properties of knots, including three-coloring of knots. A knot diagram was three-colorable if it satisfied the following properties:

1. Every arc was colored red, blue or yellow
2. Every crossing had all three colors appear, or only one color appear.
3. All colors were used

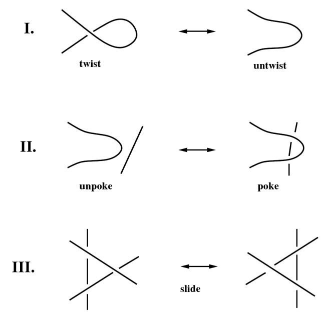


Figure 1: The three Reidemeister moves.

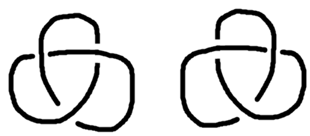


Figure 2: The Right and Left Trefoil in all of their glory.

We also showed that the Reidemeister moves were compatabile with three colorability. This invariant could tell us that the trefoil knot was not a trivial knot.

Problem 2.1. Can you find a knot that is not trivial, but not three colorable? Why does this make the three-coloring knot invariant less-than-ideal?

Problem 2.2. Suppose we loosen the definition of three coloring by dropping condition 3. We can then look at the three-coloring number, the number of ways to three-color a knot. What is the three-coloring-number of the trefoil? What about the unknot?

Problem 2.3. Compute the three-coloring number of 5_1 (drawn on the board)

Problem 2.4. We define a link to be a several closed pieces of string and we also represent it with a projection. Can you find a link that has the same three-coloring-number as the trefoil? Is this link three-colorable?

Problem 2.5. Can the three coloring number or the three coloring invariant distinguish between the right and left trefoil? (see Figure 2)

3 Bridge Number

An arc that begins at an undercrossing, goes through several overcrossings, and ends at an undercrossing is called a bridge. The Bridge number of a diagram is the number of bridges that are present in it. However, the bridge number is not necessarily the same as the number of crossings in the diagram. For instance, the bridge number of Figure 3 is 1, but the number of crossings is 5.

Problem 3.1. Which is bigger, the bridge number or the crossing number of a diagram? When are they equal?
The bridge number of a diagram, like the crossing number of a diagram, is not invariant under Reidemeister moves, so it is not such a wonderful knot invariant. We can force this property by looking at the minimal bridge number over all diagrams.
We define the Bridge Number of a knot to be the minimal bridge number of all possible diagrams.

Problem 3.2. What is the smallest possible bridge number. Can a knot with this bridge number be non-trivial?

Problem 3.3. What is the smallest possible bridge number for a non-trivial knot? How many knots have this bridge number?

Problem 3.4. Compute the bridge number for the trefoil knot.

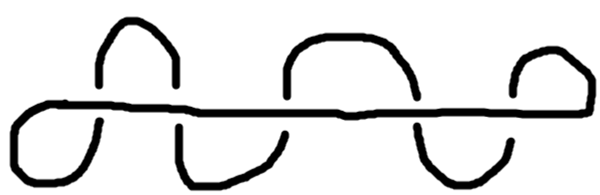
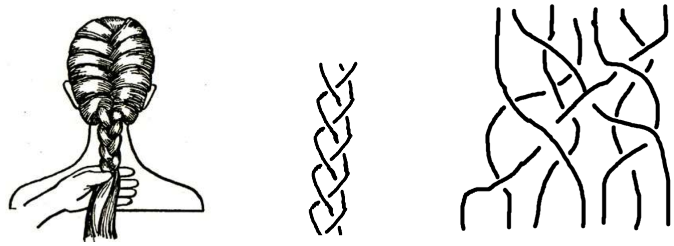


Figure 3: A poorly drawn trivial knot.



(a) Braided Hair. (b) A slightly less appealing representation of the same braid. (c) An ugly braid.

Figure 4: Some Braids of various elegance.

4 Braids

We define a braid to be a collection of strings that run from one plane to another, such that no string doubles back on itself. Braids are a lot like knots, in that we represent them with diagrams as well. However, they have a far more accessible structure, as they run only “up and down”. Both braids and knots have crossings with many of the same properties. We will, in fact, define relationship between braids using the Reidemeister moves as well.

Problem 4.1. Which of the Reidemeister moves can be applied to braids? Why not the other(s)?

Problem 4.2. Describe a braid diagram (figure 4(b)) to someone else at your table without drawing any pictures. Is this more difficult or easier than describing the structure of a knot?

Problem 4.3. Can you come up with a formal way to describe braids?

We will describe braids as elements of a group; a set where there is a “multiplication” that satisfies the usual properties. The set that we will be looking at is $\mathcal{B}_n$, the collection of all braids with n strings.

Problem 4.4. How many elements are in $\mathcal{B}_n$?

In this group we will be able to combine braids by “stacking” (figure 5). If we have two braids, β_1 and β_2, we will write the braid resulting from their stacking as $\beta_3 = \beta_1\beta_2$.

For the following problems, consider the braids in $\mathcal{B}_4$.

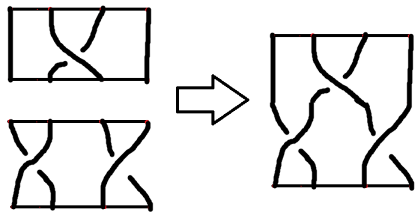


Figure 5: A crudely drawn diagram of Braid Composition

Problem 4.5. Can you think of a braid β_I such that when it is combine with any other braid β, we have that $\beta_I\beta = \beta\beta_I = \beta$?

Problem 4.6. Given β_1 can you think of a braid β_1^{-1} such that $\beta_1^{-1}\beta_1 = \beta_1\beta_1^{-1} = \beta_I$? Draw a few examples to see how it works.

Problem 4.7. Can you think of two braids β_1 and β_2 such that $\beta_1\beta_2 = \beta_2\beta_1$?

We can begin to look at all of the generators of braids. We will call them $\sigma_1, \sigma_2 \ldots \sigma_{n-1}$, and they will be the crossing of the i and $i+1$ strings. We can make any braid diagram by stringing these simplest braids together (Figure 6).

Problem 4.8. Can you represent Reidemeister's 2nd move in terms of braids?

Problem 4.9. Can you represent Reidemeister's 3rd move in terms of braids?

Problem 4.10. Do these represent all of the possible relationships between braids?

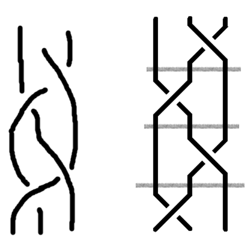


Figure 6: Braid on left represented as $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$.

5 Braid Closures

We can start to make knots out of braids. We simply glue the bottom of the braid to the top; we call this operation the closure of the braid, and represent it as $\bar{\beta}$. The simplest way to understand this is to see a picture; the closure $\bar{\sigma_1^3}$ from the braid group $\mathcal{B}_2$ is shown in Figure 7. This gives us two new important invariants; the Braid index and Braid Word. We define the Braid index of a knot K to be the smallest n such that the closure of a braid from $\mathcal{B}_n$ is K. We then define the Braid Word to be a braid of shortest length (meaning having the least number of symbols) whose closure is K.

Problem 5.1. Is the Braid Index a good invariant? In particular, are their multiple different knots with the same braid index? Can a knot have multiple different values for the Braid Index?

Problem 5.2. How is the length of the Braid Word related to it's knot?

Problem 5.3. Use the braid index to prove that the right trefoil is not the same as the left trefoil.

Problem 5.4. What knots have braid index 2?

Problem 5.5. Where have we seen the braid index before?

6 Things to Think About

Problem 6.1. We can use our Braid Relationships to resolve Reidemeister's 2nd and 3rd moves in our braid word; is there a way to represent the 1st one?

Problem 6.2. Under what conditions is the closure of a braid a knot? A link on 2 strings? A link on n-strings?

Problem 6.3. It is easy to see that the closure of every braid is either a knot or a link. Is their a braid to represent every single knot or link?

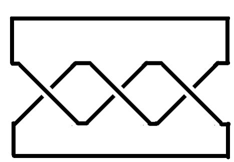


Figure 7: The closure of σ_1^3.