Introduction to Groups II

Matthew Gherman and Adam Lott

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Notation. Let $(G, *)$ be a group and let $a, b \in G$. From now on, we will drop the symbol $*$ and simply write ab to mean $a * b$. We will refer to the operation as "multiplication" (even though it isn't literally multiplication in the usual sense).

1 Group morphisms

Until now, we have only studied individual groups by themselves. Now we want to expand our horizons and study how different groups relate to each other.

Definition 1. Let G and H be two groups and let $\phi : G \to H$ be a function. The function ϕ is said to be a *group morphism* if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$.

Exercise 1. Determine whether or not each of the following is a group morphism and PROVE YOUR ANSWERS.

(a) $G = (\mathbb{Z}, +)$, $H = (\mathbb{Z}/20\mathbb{Z}, +)$, $\phi(x) = x \bmod 20$.

(b) $G = H = (\mathbb{R}, +)$, $\phi(x) = x + 1$.

(c) $G = H = (\mathbb{R} \setminus \{0\}, \cdot)$, $\phi(x) = x^2$.

(d) $G = (\mathbb{Z}/4\mathbb{Z}, +)$, $H = (\mathbb{Z}/8\mathbb{Z}, +)$, $\phi(x) = 2x \bmod 8$.

(e) $G = (\mathbb{Z}, +)$, $H = (\mathbb{R}, +)$, $\phi(x) = 0$ for all x.

(f) $G = (\mathbb{R}, +)$, $H = (\mathbb{R}^+, \cdot)$, $\phi(x) = 2^x$.

(g) $G = (\mathbb{Z}/10\mathbb{Z}, +)$, $H = (\mathbb{Z}/3\mathbb{Z}, +)$, $\phi(x) = x \bmod 3$.

(h) (CHALLENGE). Let G be the set of 2×2 matrices of the form $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$ where $x, y \in \mathbb{R}$ and x and y are not both 0. This set forms a group under the operation of matrix multiplication (you can assume this). Let H be the set of nonzero complex numbers with the operation of multiplication. Define $\phi : G \to H$ by $\phi\left(\begin{bmatrix} x & y \\ -y & x \end{bmatrix}\right) = x+iy$.

Exercise 2. Let $\phi : G \to H$ be a group morphism.

(a) Prove that $\phi(e) = e$.

(b) For any $g \in G$, prove that $\phi(g)^{-1} = \phi(g^{-1})$.

Definition 2. Let $\phi : G \to H$ be a group morphism. The *kernel* of ϕ is defined by $\ker(\phi) := \{g \in G : \phi(g) = e\} \subseteq G$, i.e. $\ker(\phi)$ is the set of elements of G that get mapped to the identity element of H by ϕ.

The *image* or *range* of ϕ is defined as $\mathrm{Im}(\phi) = \{\phi(g) : g \in G\} \subseteq H$, i.e. $\mathrm{Im}(\phi)$ is the set of all elements in H which are equal to ϕ(something in G).

Exercise 3. For each of the following group morphisms, list its kernel and its image.

(a) $G = (\mathbb{Z}, +)$, $H = (\mathbb{Z}/20\mathbb{Z}, +)$, $\phi(x) = x \bmod 20$.

(b) $G = (\mathbb{R}, +)$, $H = (\mathbb{R}^+, \cdot)$, $\phi(x) = 2^x$.

(c) $G = (\mathbb{Z}/4\mathbb{Z}, +)$, $H = (\mathbb{Z}/8\mathbb{Z}, +)$, $\phi(x) = 2x \bmod 8$.

(d) $G = (\mathbb{Z}, +)$, $H = (\mathbb{R}, +)$, $\phi(x) = 0$ for all x.

There is a certain kind of special group morphism that gets its own name:

Definition 3. A group morphism $\phi : G \to H$ is said to be an *isomorphism* if ϕ is both one-to-one and onto. Two groups G and H are said to be *isomorphic* if there exists an isomorphism between them.

Exercise 4. For each of the given group morphisms, decide whether or not it is an isomorphism and PROVE YOUR ANSWERS.

(a) $G = (\mathbb{Z}, +)$, $H = (\mathbb{Z}/20\mathbb{Z}, +)$, $\phi(x) = x \bmod 20$.

(b) $G = H = (\mathbb{Z}/7\mathbb{Z}, +)$, $\phi(x) = 2x \bmod 7$.

(c) $G = (\mathbb{R}, +)$, $H = (\mathbb{R}^+, \cdot)$, $\phi(x) = 2^x$

(d) $G = (\mathbb{Z}/4\mathbb{Z}, +)$, $H = (\mathbb{Z}/8\mathbb{Z}, +)$, $\phi(x) = 2x \bmod 8$.

Exercise 5. In group theory, we often consider two isomorphic groups to be the "same" group. Think about why this is and write a few sentences explaining your thoughts.

Exercise 6. (CHALLENGE) Let G be a cyclic group with $|G| = n$. Prove that G is isomorphic to $(\mathbb{Z}/n\mathbb{Z}, +)$.

Exercise 7. (SUPER CHALLENGE) Suppose G and H are finite groups. Let $\phi : G \to H$ be a group morphism. Prove that $|\operatorname{Im}(\phi)| \cdot |\ker(\phi)| = |G|$.

2 Cosets and normal subgroups

Definition 4. Let H be a subgroup of a group G. Define the set $aH = \{ah : h \in H\}$ (or sometimes $a + H = \{a + h : h \in H\}$ if our operation is addition). This is referred to as a *left coset* of H. The set of left cosets of H, denoted G/H, is the set $\{aH : a \in G\}$.

Exercise 8. Write down a similar definition for a right coset of a subgroup H of a group G. Which property of G guarantees left and right cosets will coincide?

Exercise 9. (a) Find a left coset of $3\mathbb{Z}/6\mathbb{Z}$ in the group $(\mathbb{Z}/6\mathbb{Z}, +)$.

(b) Describe a left coset of $\pi\mathbb{R}$ in the the real numbers under addition $(\mathbb{R}, +)$?

(c) Take the group $(\mathbb{Z}, +)$, and list the cosets of $4\mathbb{Z} = \{n \in \mathbb{Z} : n = 4k \text{ for some } k \in \mathbb{Z}\}$, the multiples of 4.

Exercise 10. Let H be a subgroup of a group G.

(a) Denote by e the identity element of G. Show that $gH = H$ for $g \in G$ if and only if $g \in H$.

(b) Prove that every group element $a \in G$ appears in one and only one left coset of H.

Exercise 11. Let G be a finite group.

(a) Show that any two left cosets of a subgroup H in G have the same number of elements. (Hint: It is enough to show that each aH has the same number of elements as H.)

(b) (Lagrange's Theorem) Prove that the number of elements of H divides the number of elements of G.

(c) (CHALLENGE for budding set theorists) Do we need the finite condition of G for (a) if we compare the cardinalities of two left cosets?

Definition 5. A *normal subgroup* H of G is a subgroup such that $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.

Exercise 12. (a) Is $4\mathbb{Z}$ in $(\mathbb{Z}, +)$ a normal subgroup?

(b) As discussed last week, let D_4 be the group of symmetries of a square in the plane. Let r denote the counter-clockwise rotation by 90° and s denote reflection about a vertical line through the middle. All elements of D_4 are listed $\{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$ where sr represents the rotation r followed by the reflection s. Show that $\{e, r, r^2, r^3\}$ is a normal subgroup of D_4. Show that $\{e, s\}$ is not a normal subgroup of D_4. Is $\{e, r^2\}$ a normal subgroup of D_4?

(c) Is $\mathbb{Z}$ a normal subgroup of $(\mathbb{R}, +)$?

(d) Is $\mathbb{Q}^+$ a normal subgroup of $(\mathbb{R}^+, \times)$?

Exercise 13. Let G be an abelian group. As a result, we will write the operation $+$.

(a) Show that every subgroup of G is normal.

(b) Show that the left coset $a + H$ of H in G is the same as the right coset $H + a$ as sets. At long last, this statement answers the question posed in Exercise 8.

Exercise 14. Let G be a finite group and H a subgroup such that there are two left cosets of H in G. Prove that H is normal in G.

Exercise 15. Let $\phi : G \to H$ be a group morphism.

(a) Prove that $\ker(\phi)$ is a subgroup of G.

(b) Prove that $\ker(\phi)$ is a *normal* subgroup of G.

(c) Prove that $\mathrm{Im}(\phi)$ is a subgroup of H.

(d) (CHALLENGE). Give an example to show that $\mathrm{Im}(\phi)$ is not necessarily a normal subgroup of H.

Exercise 16. (CHALLENGE) True or false: there exists a group G and a *proper* subgroup H (meaning $H \subsetneq G$) such that G is isomorphic to H.

3 Quotient groups

And you may ask yourself, "Why have we have defined all of this random stuff?" It turns out that studying the quotients of groups and morphisms between groups allows us to more fully understand groups. Okay, fair, if you do not care about groups then this is not a great answer.

Definition 6. Let G be a group and H be a normal subgroup. Then we can define multiplication of two left cosets $aH, bH \in G/H$ by $aH \cdot bH = (ab)H$. We call G/H a *quotient group*.

Exercise 17. (a) Write down $\mathbb{Z}/4\mathbb{Z}$ and describe the group operation.

(b) In Exercise 12(b), we hopefully showed that $H = \{e, r^2\}$ is a normal subgroup. Write down D_4/H and describe the group operation.

(c) Write down $\mathbb{R}/\mathbb{Z}$ and describe the group operation.

(d) (CHALLENGE) Write down $\mathbb{R}^+/\mathbb{Q}^+$ (where the operation on $\mathbb{R}^+$ is multiplication) and describe the group operation on the quotient group.

Exercise 18. (a) Show that the multiplication in G/H is well-defined. In other words, take $aH, a'H \in G/H$ such that $aH = a'H$. Show that $aH \cdot bH = a'H \cdot bH$. Remember that H is normal! We have shown that the multiplication does not depend on how we represent the left coset.

(b) Show that G/H with the multiplication defined satisfies the axioms of a group.

Exercise 19. Review Exercise 12(b). Show that the multiplication defined on $S_3/\{e, s\}$ does not satisfy the group axioms. It turns out that being normal is exactly the condition a subgroup H needs to satisfy so that multiplication in G/H can be defined.

4 The First Isomorphism Theorem (OPTIONAL)

In this section, we will outline a proof of one of the most foundational theorems in all of group theory.

Theorem 1 (First Isomorphism Theorem)**.** *Let $\phi : G \to H$ be a group morphism. Recall that $\ker(\phi)$ is a normal subgroup of G and $\mathrm{Im}(\phi)$ is a subgroup of H. Then $G/\ker(\phi)$ is isomorphic to $\mathrm{Im}(\phi)$.*

Exercise 20. Prove Theorem 1.

(a) Explain why we can regard ϕ as a morphism $G \to \mathrm{Im}(\phi)$ instead of as a morphism $G \to H$.

(b) Define a new map $\pi : G \to G/\ker(\phi)$ by $\pi(g) :=$ the coset $g \cdot \ker(\phi)$. Prove π is a group morphism.

(c) Define a new map $f : G/\ker(\phi) \to \mathrm{Im}(\phi)$ by $f(g \cdot \ker(\phi)) = \phi(g)$. Prove that f is well-defined, i.e. that if $g \cdot \ker(\phi) = g' \cdot \ker(\phi)$, then $f(g) = f(g')$.

(d) Prove that f is a group morphism.

(e) Prove that f is an isomorphism.