

EXPECTED NUMBER OF TEARS AND TANTRUMS

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1 *A useful estimate.* Let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Why is H_n approximately equal to $\log(n)$, where the logarithm base is $e \approx 2.71828$? NO CALCULUS, PLEASE!

2 *A true story.* When I first organized the Bay Area Mathematical Olympiad, I needed to send registration forms out with random ID numbers for participants. So I made a list of the numbers from 1 to 1000, and then used my sampling software to take a random sample of size 1000 from these numbers. However, I stupidly forgot to check the “sample without replacement” button and instead I sampled *with* replacement. How many distinct ID numbers were produced?

3 *A true story.* In the SF Math Circle for elementary school kids, 11 8-year-old kids stood in a circle. They wrote their names on a piece of paper, and the instructor put them in a box and shook the box. Then each kid randomly chose a name. The instructor handed a kid a beanbag ball and the kid then tossed the ball to the person whose name they had. And so it continued. If not all kids got a ball tossed to them, the instructor gave the ball to one of those left-out kids and the process continued.

- (a) If a kid ended up tossing a ball to him or herself, that kid cried. On average, how many kids cry?
- (b) What is the probability that no kids cry?
- (c) The instructor wanted all the kids to be able to toss the ball without intervention. In other words, ideally, all 11 kids will form a “cycle.” What is the probability that this happens?
- (d) If all the kids are not in one cycle, the instructor asked the kids to change names so that this can be achieved. To keep anarchy at bay, the instructor only allowed two kids at a time to exchange their slips of paper. On average, how many such exchanges are needed?
- (e) Another desirable scenario for the instructor was for a majority of the kids to be in a cycle. Otherwise, kids have tantrums. What’s the probability of a tantrum?

4 *Snake necklaces.* Imagine a pit of 100 snakes, and James Bond is thrown into the pit. He fearlessly, and randomly, grabs ends of snakes (ignoring whether it is the head or tail) and deftly ties them together. He keeps doing this until he is left with a bunch of “snake necklaces” that cannot harm him. How many necklaces will there be?