Math Circle
Beginners Group
January 10, 2016
Geometry I

Warm-up Problem

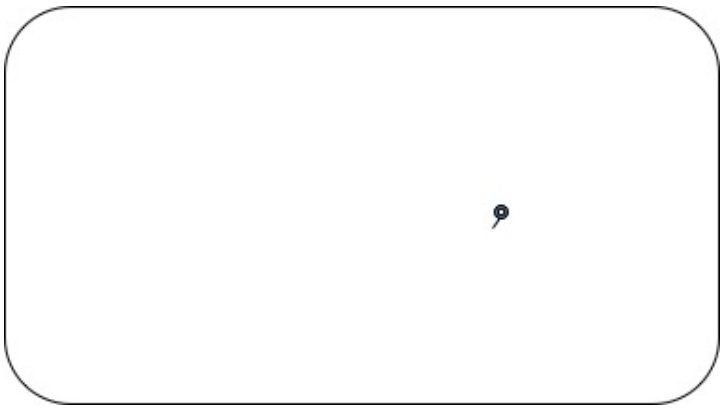
A bridge is on the verge of collapse. Four people want to cross the bridge to reach a safer area before it collapses. It is a dark, stormy night, and there is only one flashlight available among them. The flashlight is required to make a trip across the bridge. The bridge can only take the weight of two people at a time. Alan takes a minute to cross the bridge. Ben takes 2 minutes. Cathy takes 5 minutes. Das takes 10 minutes to cross the bridge. What is the shortest time the four people will take to cross the bridge, and how?

Tying Goats[1]

This problem set deals with goats. Goats are ravenous and consume everything they can reach. Therefore, they are usually kept on a rope attached to a stake planted in the pasture.

1. Imagine that in the field below, a goat is attached to the given stake with a rope of length r.

   (a) Draw the section of the pasture that can be consumed by the goat.

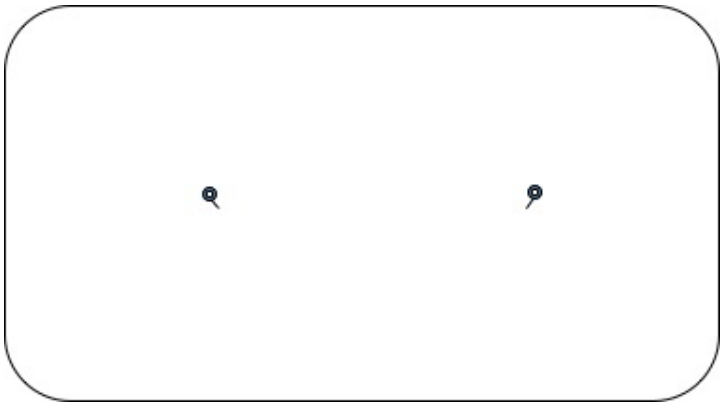


[1]The problems are taken from "A Moscow Math Circle: Week-by-week Problem Sets" by Sergey Dorichenko.

(b) What is the area (in terms of r) of the section the goat can consume?

2. Now imagine that a rope of length l has been stretched between two stakes in a field. A goat is tied to this rope with another rope of length r using a ring that is free to slide along the first rope.

   (a) Draw the section of the pasture that can be consumed by the goat.

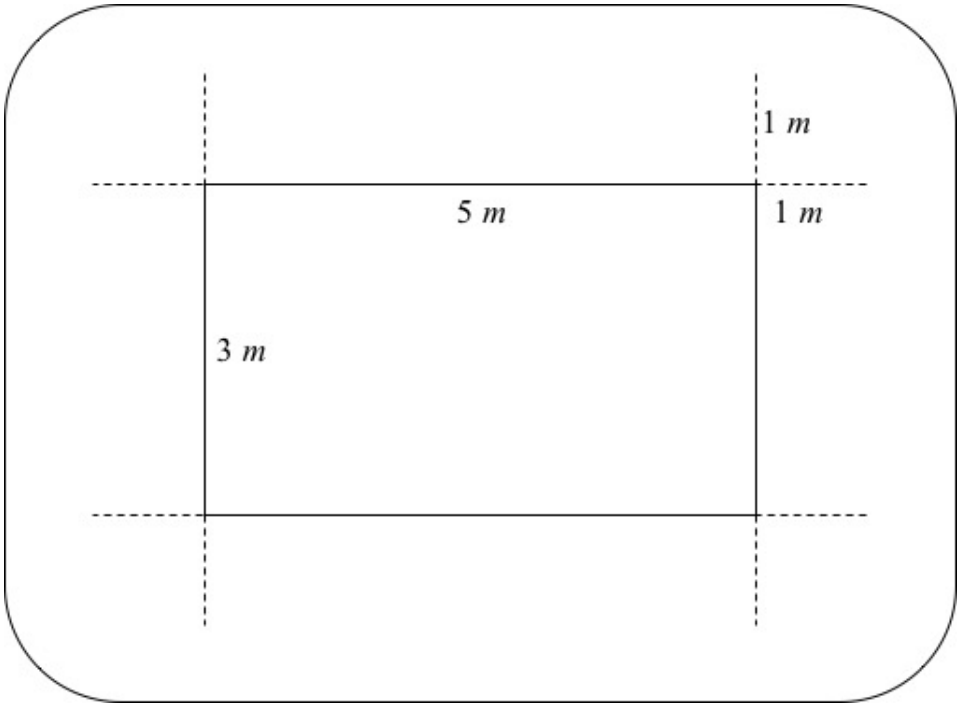


   (b) What is the area (in terms of l and r) of the section the goat can consume?

These two shapes were simple to construct. So we can call the circle and the stadium shapes that you obtained above our "basic shapes." Let us try to use these shapes for more complicated setups of ropes.

3. A mathematician took a walk on a field holding a goat on a 1-meter-long rope. The mathematician's path was rectangular with dimensions 3 meters by 5 meters. Draw the section of the field the goat will have consumed by the end of the mathematician's walk.

   (a) Draw the section of the pasture that can be consumed by the goat.



   (b) What is the area, expressed in square meters, of the section the goat can consume?

What if we want to constrain the area that the goat can consume to a semi-circle, or any other shape?

We would have to use more than one rope of different lengths. This way, we can have basic shapes of different sizes, i.e. circles of different radii and stadiums of different dimensions. The careful intersection of these basic shapes will yield the shape that we desire.

Suppose we know how to tie the goat so that it can only eat the grass in a particular basic shape X, and we also know how to tie the goat so that it can only eat the grass in a basic shape Y. If we want the goat to eat only the grass in the intersection of the shapes X and Y, we should tie the goat in both ways.

4. How can a goat be constrained to consume grass in the shape of

   (a) a semi-circle of radius r?

      i. How do two basic shapes intersect to form a semi-circle? Show the intersection below. What are the lengths of the ropes that you will use?

      ii. What is the area of the pasture that the goat can consume? The answer should be in terms of r.

   (b) a square of side length r?

      i. How do two basic shapes intersect to form a square? Show the intersection below. What are the lengths of the ropes that you will use?

ii. What is the area of the pasture that the goat can consume? The answer should be in terms of r.

(c) a rectangle of length l and width r?

i. How do two basic shapes intersect to form a rectangle? Show the intersection below. What are the lengths of the ropes that you will use?

ii. What is the area of the pasture that the goat can consume? The answer should be in terms of l and r.

(d) an equilateral triangle of side length r?

i. How do three basic shapes intersect to form a triangle? Show the intersection below. What are the lengths of the ropes that you will use?

ii. What is the area of the pasture that the goat can consume? The answer should be in terms of r.

5. Dogs can be used to herd goats because a goat will not occupy a space that a dog can reach. However, do not let a dog run free since it will chase the goat constantly and never let the goat rest or eat.

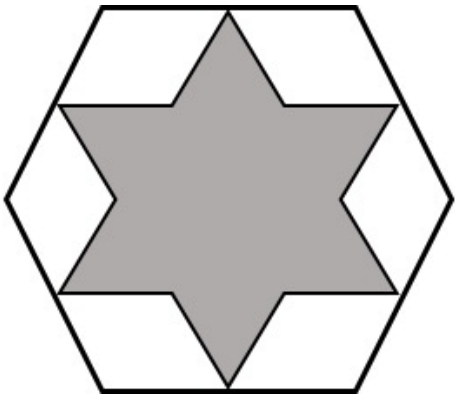
   (a) How can a dog hold a goat in the shape of a ring? Draw the setup below.

   (b) How can a dog hold a goat in the shape of a semi-circle? Draw the setup below.

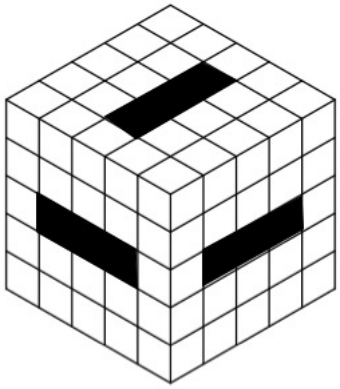
   (c) Using tied dogs, contain an untied goat in the shape of a triangle. Draw the setup below.

Math Kangaroo[2]

1. The vertices of the star shown in the picture are the midpoints of the sides of a regular hexagon. If the area of the star is 6, what is the area of the hexagon?

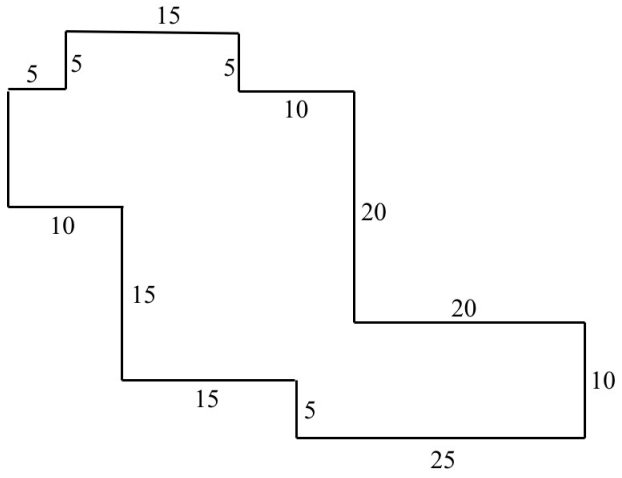


2. The cube shown is made out of small cubes. Inside the big cube, tunnels were made going through the cube in such a way that they are parallel to the walls of the cube (see picture). After making the tunnels, how many small cubes are left in the solid?

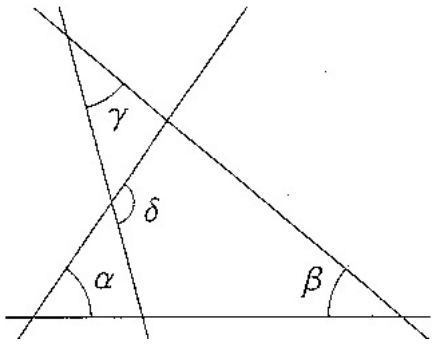


[2]The problems are taken from Math Kangaroo contests from the years 2001, 2002, 2011 and 2013.

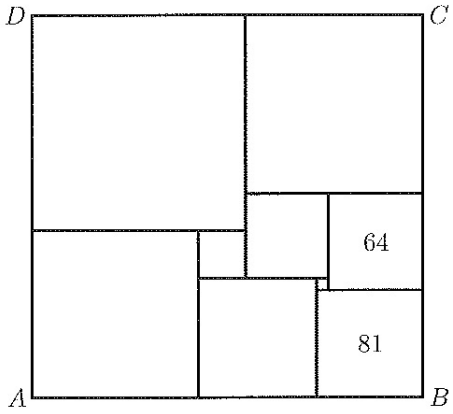
3. Janusz's garden is shaped in the way shown in the picture. The lengths of its sides are given in meters, and any two adjacent sides are perpendicular. What is the area of the garden in square meters?



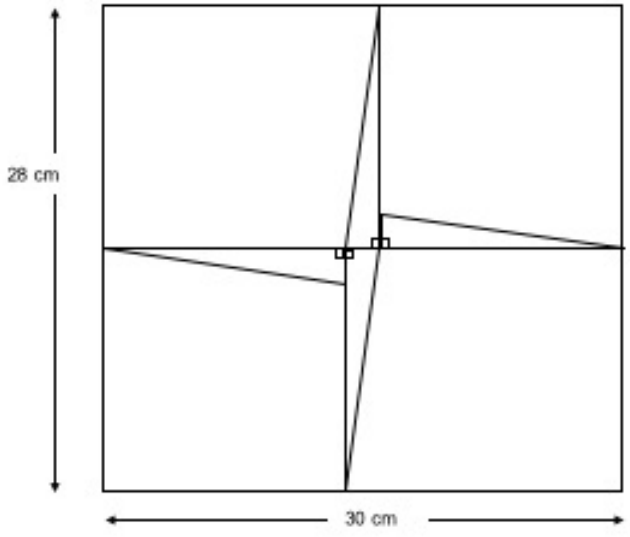
4. In the diagram, $\alpha = 55°$, $\beta = 40°$, and $\gamma = 35°$. What is the measure of δ?



5. Rectangle $ABCD$ is divided into 9 squares. The areas of two of the squares are $64in^2$ and $81in^2$, as shown in the picture. What is the length of the side AB?



6. There are four identical right triangles inside a rectangle, as shown in the picture. The lengths of the two sides of the rectangle are $28cm$ and $30cm$. What is the sum of the areas of all four triangles?



7. A square piece of paper was cut into six rectangles as shown. The sum of the perimeters of these six rectangles is equal to $120cm$. What is the area of the sheet of paper?