

# Just how big are big numbers? Bigger than you think!

Math Circle

4/7/2019

I would like to start off this week by doing something that **has never been done before in the history of humanity**. Please, direct your attention to the board.

1. Before we move on, let us do a brief review of laws of exponents. For all of the following problems you don't have to compute the exact decimal number. You just have to simplify the algebraic expression.

(a) Can you simplify  $2^{13} \cdot 2^7$ ? No need to compute the exact value, just simplify it into one expression.

(b) Can you simplify  $3^8 \cdot 7^8$ ?

(c) Can you simplify  $11^{13} \cdot 17^{19}$ ?

(d) Can you simplify  $2^{13} \cdot 8^{-4}$ ?

(e) Please simplify  $2^{(3^4)}$  and  $(2^3)^4$ . Note that these are **not** the same thing

2. A very useful function that you might not have heard of before is called the logarithm (usually shortened to just log). The log is defined as follows:

$$\log_a(x) = b \text{ is the same thing as } a^b = x \text{ as long as } a > 0.$$

So, for example,

$$\log_2(8) = 3 \text{ because } 2^3 = 8$$

$$\log_{10}(100) = 2 \text{ because } 10^2 = 100$$

$$\log_9(3) = 1/2 \text{ because } 9^{1/2} = 3$$

$$\log_4(1/16) = -2 \text{ because } 4^{-2} = 1/16.$$

The expression  $\log_a(x) = b$  is read "The logarithm of  $x$  in the base  $a$  equals  $b$ ." The logarithm might look strange but trust me, it is very useful!

(a) What's the value of  $\log_6(36)$

(b) What's the value of  $\log_5(125)$

(c) What's the value of  $\log_{81}(3)$

(d) What's the value of  $\log_{12395817}(12395817)$ ? This one looks hard, but is actually quite easy!

(e) Suppose that  $\log_{131}(x) = 0$ . What is  $x$ ?

(f) What is  $\log_7(7^{65})$ ? This question is either very easy or very difficult. Check with your instructor if you get stuck.

(g) What are the values of  $\log_5(1/5)$ ,  $\log_5(0)$ , and  $\log_5(-5)$ ?

3. Logarithms and exponents are very closely related (just like addition/subtraction and multiplication/division). For every law of exponents there is an equivalent law of logarithms.

For example think about the law  $a^b \cdot a^c = a^{b+c}$ . If we notate  $a^b$  as  $x$  and  $a^c$  as  $y$ , then we get:

$$\begin{aligned}a^b &= x \\a^c &= y \\a^{b+c} &= xy\end{aligned}$$

By converting the above into logarithms we get that:

$$\begin{aligned}\log_a(x) &= b \\ \log_a(y) &= c \\ \log_a(xy) &= b + c\end{aligned}$$

Therefore, we can infer that:

$$\log_a(x) + \log_a(y) = \log_a(xy)$$

and voilà! We proved one of the laws of logarithms! Notice that our final answer is just a statement about logarithms, and exponents are nowhere to be seen.

- (a) Find the logarithm law that corresponds to the law of exponents,  $(a^b)^c = a^{b \cdot c}$ . Have an instructor check your answer. \*Hint, let  $x = a^b$ .

- (b) Prove the change of base formula for logarithms, that is that

$$\frac{\log_a(x)}{\log_a(b)} = \log_b(x).$$

- (c) Using the above laws, prove that exponentiation base  $a$  and logarithms base  $a$  are inverses of each other. In other words, prove that  $a^{\log_a(x)} = x$  and also that  $\log_a(a^x) = x$ .

4. A common convention is that a log without a base is  $\log_{10}$ . From now on, if you see a logarithm without a base, you can assume that the base is base 10. Please solve the following log questions!

(a) What are the values of the following?

$$\begin{aligned} \log(10) &= \\ \log(100) &= \\ \log(1,000) &= \\ \log(1,000,000) &= \\ \log(0.0001) &= \end{aligned}$$

(b) Mathematicians say that the logarithm is monotone increasing. This sounds complicated, but really it just means that the larger number you put into it, the larger number you get out. In mathematical terms, if  $a > b$  then  $\log(a) > \log(b)$  as well.

The number  $\log(365)$  is a whole number plus an infinite decimal. Can you compute the integer part? \*Hint, use your answers from the last question.

(c) The company Google is named after the number googol, which is defined as 1 with one hundred zeros after it. What is  $\log(\text{googol})$ ? A googolplex =  $10^{\text{googol}}$ . What is  $\log(\text{googolplex})$ ?

(d) Lastly, below is a partially finished table of logarithms. Can you please fill in the missing values? You'll need this table for the rest of the handout.

$x$	$\approx \log(x)$	$x$	$\approx \log(x)$	$x$	$\approx \log(x)$
1	0	6	-----	11	1.04
2	0.30	7	0.85	12	-----
3	0.48	8	-----	13	1.11
4	-----	9	-----	14	-----
5	0.70	10	-----	15	-----

5. Now let us return to the main thrust of the handout. Just how large is  $52!$  really?

(a) As a little warm-up, please compute  $1!$ ,  $2!$ ,  $3!$ ,  $4!$  and  $5!$ .

(b) Please rate the following numbers from smallest to largest:

$$3 \cdot 6, 6!, 3^6, 6^3$$

(c) Do the same for the following. Form an educated guess, but do not actually compute.

$$52!, 2^{200}, 25^{52}, 52^{10}$$

(d) It is actually pretty difficult to estimate the relative sizes of factorials and exponents, but there is a secret. Don't compare the numbers directly. Instead compare their logarithms. Approximate

$$\log(2^{200}), \log(25^{52}), \text{ and } \log(52^{10})$$

by using the table that you computed in question 4.d)

- (e) Please show that  $\log(6!) = \log(1) + \log(2) + \log(3) + \log(4) + \log(5) + \log(6)$ . Then, compute  $\log(6!)$  using the logarithm table.

- (f) There is a formula called Stirling's formula that approximates  $\log(n!)$ . The formula is:

$$\log(n!) \approx n \cdot \log(n) - 0.43 \cdot n + 0.5 \cdot \log(n) + 0.4$$

Use Stirling's formula to approximate  $\log(6!)$ . Compare this to your answer to the previous question.

- (g) Use Stirling's formula to approximate  $\log(52!)$ .

- (h) How many digits before the decimal does  $52!$  have? Use your answers from 5.d) and 5.g) to finally give a rigorous answer to 5.c) How close was your guess?
6. Now that we have a full command of Stirling's formula and logarithms, let's finally talk about why I can be so sure that the trick that I did at the start of class has never been done before.
- (a) Suppose that someone spent their whole life shuffling cards so that they could see as many shuffles as possible. If they lived for 100 years, and shuffled a deck of cards once per minute how many shuffles would they have done? A year is 525,600 minutes.
- (b) Is there even a 1% chance that they had seen our shuffling before? Why or why not?
- (c) What about there was a country of 1 billion ( $10^9$ ) people who all shuffled cards all day for 100 years. Would they have a 1% chance of seeing our sequence of cards?



(d) Current estimates are that there are about 100 billion people who have ever lived. If for all of human existence we lived for 100 years and did nothing other than shuffle cards, what is the likelihood that one person would have seen our sequence?

(e) Do you now believe me when I can say with confidence that no one has done exactly what we did at the start of class before? Why or why not?

7. Here are some more problems, that showcase just how big  $52!$  really is...

(a) Let's pretend that you were looking to kill some time. Every second you take a drop of water out of the ocean and put it aside in a (very large) bucket. When all of the water on all of the oceans is in the bucket, you refill the oceans and place one grain of sand on the ground. You repeat this process and place a second grain of sand on top of the first, then you repeat it a third time, etc... until you make a pile of sand high enough to touch the moon. Would you have waited more or less than  $52!$  seconds? The ocean has about  $2.6 \cdot 10^{25}$  drops in it, a grain of sand is at smallest  $4 \cdot 10^{-6}$  meters in diameter and the moon is about  $3.8 \cdot 10^8$  meters away.

(b) As a way to kill a little more time, suppose that every time your sand pile reached the moon, you put all of the sand back and bought a power ball (the largest US lottery) ticket. Every time you won the lottery you rolled 10 dice. You did this process until you all 10 dice came up 6. How long would you have waited? More than  $52!$  seconds or not? The chances of winning powerball are about 1 in 300 million, and  $6^{10}$  is about  $6 \cdot 10^7$ .