Continued Fractions I

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Definition 1. A *finite continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}}}$$

where $a_0, a_1, \ldots, a_k$ are natural numbers. It is also denoted $[a_0, a_1, \ldots, a_k]$.

Exercise 1. Write each of the following as a continued fraction:
(a) $5/12$ (b) $5/3$ (c) $33/23$ (d) $37/31$

Exercise 2. Write each of the following continued fractions as a regular fraction in lowest terms:
(a)$[2,3,2]$ (b) $[1,4,6,4]$ (c) $[2,3,2,3]$ (d) $[9,12,21,2]$

Exercise 3. Let p/q be a positive rational number in lowest terms. Perform the Euclidean algorithm to obtain the following sequence:

$$\begin{aligned} p &= q_0 q + r_1 \\ q &= q_1 r_1 + r_2 \\ r_1 &= q_2 r_2 + r_3 \\ &\vdots \\ r_{k-1} &= q_k r_k + 1 \\ r_k &= q_{k+1} \end{aligned}$$

(We know that we will eventually get 1 as the remainder because p and q are relatively prime). Prove that $p/q = [q_0, q_1, \ldots, q_{k+1}]$.

Exercise 4. Repeat Exercise 1 using the method outlined in Exercise 3.

Definition 2. Continued fractions don't always have to be finite. An *infinite continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ldots}}}}$$

where $a_0, a_1, a_2, \ldots$ are natural numbers. To prove that this expression actually makes sense and equals a finite number is beyond the scope of this worksheet, so we assume it for now. This is denoted $[a_0, a_1, a_2, \ldots]$.

Exercise 5. Using a calculator, compute the first five terms of the continued fraction expansion of the following numbers:
(a) $\sqrt{2}$ (b) π ($\approx 3.14159\ldots$) (c)$\sqrt{5}$ (d) e ($\approx 2.71828\ldots$)

Do you notice any patterns?

Exercise 6. Let α be a positive real number. Prove that α can be written as a finite continued fraction if and only if α is rational. (Hint: for one of the directions, use Exercise 3).

Definition 3. The continued fraction $[a_0, a_1, a_2, \ldots]$ is *periodic* if it is of the form

$$[a_0, \ldots, a_r, a_{r+1}, \ldots, a_{r+p}, a_{r+1}, \ldots, a_{r+p}, a_{r+1}, \ldots, a_{r+p}, \ldots].$$

In this case it is denoted $[a_0, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+p}}]$.

Example 1.
(a) $[1, 2, 2, 2, \ldots] = [1, \overline{2}]$ is periodic.
(b) $[1, 2, 3, 4, 5, \ldots]$ is *not* periodic.
(c) $[1, 3, 7, 6, 4, 3, 4, 3, 4, 3, \ldots] = [1, 3, 7, 6, \overline{4, 3}]$ is periodic.
(d) $[1, 2, 4, 8, 16, \ldots]$ is *not* periodic.

Exercise 7.
(a) Prove that $\sqrt{2} = [1, \overline{2}]$.
(b) Prove that $\sqrt{5} = [2, \overline{4}]$.
(Hint: use the same strategy as Exercise 5 but without a calculator.)

Exercise 8 (CHALLENGE)**.** Express the following continued fractions in the form $\frac{a+\sqrt{b}}{c}$ where a, b, and c are integers:
(a) $[\overline{1}]$ (b) $[\overline{2, 5}]$ (c) $[1, 3, \overline{2, 3}]$

Exercise 9 (EXTRA CHALLENGE)**.** Let $\alpha = [a_0, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+p}}]$ be any periodic continued fraction. Prove that α is of the form $\frac{a+\sqrt{b}}{c}$ for some integers a, b, c where b is not a perfect square.

Exercise 10 (SUPER DUPER CHALLENGE)**.** Prove that any number of the form $\frac{a+\sqrt{b}}{c}$ where a, b, c are integers and b is not a perfect square can be written as a periodic continued fraction.

Remark 1. Numbers of the form $\frac{a+\sqrt{b}}{c}$ where a, b, c are integers and b is not a perfect square are the "simplest" irrational numbers in the following sense. A number is rational if and only if it is the solution to a degree 1 polynomial equation, $ax + b = 0$. Similarly, a number is of the form $\frac{a+\sqrt{b}}{c}$ if it is the solution to a degree 2 polynomial equation, $ax^2 + bx + c = 0$ (Bonus exercise: prove this). Such numbers are called *quadratic* irrational numbers or *degree 2* irrational numbers.

Remark 2. Notice that the results of this worksheet provide a very clean characterization of continued fraction expansions:

- α is a rational number if and only if it has a finite continued fraction expansion.
- α is a degree 2 irrational number if and only if it has an infinite periodic continued fraction expansion.