Lesson 8: Similar Triangles

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Definition 1.
Two triangles are called *similar* if they have the same angles.

Theorem 1.
Triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if and only if

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$$

In other words, they have the same ratio of sides. This common ratio is called the *similarity ratio.*

Theorem 2.
Triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if and only if $\angle BAC = \angle B'A'C'$ and

$$\frac{AB}{A'B'} = \frac{AC}{A'C'}$$

In other words, they share an angle and have the same ratio of sides adjacent to that angle.

Problem 1.
In lecture we proved theorem 1. Now you need to prove theorem 2. Here are the suggested steps. If the triangles are already similar, you may use the already proven theorem 1 or the intercept theorem to get the condition on the ratios of sides. The more difficult direction is to show that if $\angle BAC = \angle B'A'C'$ and

$$\frac{AB}{A'B'} = \frac{AC}{A'C'}$$

then the triangles are similar. To show this, you should mimic the proof of theorem 1. More concretely, consider consider the triangle $\triangle ABC$ and mark the point M on the ray AB such that $AM = A'B'$. Then draw a line through M parallel to BC, and let N be the intersection of this line with the ray AC. Show that $\triangle ABC \sim \triangle AMN$, then use the intercept theorem to show that $AN = A'C'$. From there deduce that $\triangle AMN = \triangle A'B'C'$, and conclude the proof.

Problem 2.
In this problem you may use both theorem 1 and theorem 2, even if you did not solve problem 1.
a) Suppose $\triangle ABC$ and $\triangle A'B'C'$ are similar. Let AD and $A'D'$ be the angle bisectors of each triangle. Show that

$$\frac{AD}{A'D'} = \frac{AB}{A'B'}$$

Hint: Show that $\triangle BAD \sim \triangle B'A'D'$.

b) In the same setup, let AM and $A'M'$ be the medians of each triangle. Show that

$$\frac{AM}{A'M'} = \frac{AB}{A'B'}$$

Problem 3.
In a round-robin tournament with four teams you get 2 points for winning, 1 point for a draw and 0 points for losing. If team A had 5 points, team B had 2 points and team C had 1 point, which place did team D get?

Problem 4.
Let n be a positive integer divisible by 500. Show that the sum of even divisors of n is greater than the sum of odd divisors of n.

Problem 5.
Ten positive integers are written on the board, and it turns out that all their last digits are distinct and all their second-to-last digits are also distinct. Show that their sum cannot be a perfect square.