## 12 Balance scale and binary numbers

*Materials for the lesson and homework:* a couple of regular pencils, an eraser, a pencil sharpener; an abacus.

### Warm-up

**Problem 12.1** *A cutlet has two sides to fry. A frying pan is large enough to fry two cutlets at the same time, but not large enough for three. Each side of a cutlet takes one minute to fry. What is the minimal time one needs to fry three cutlets?*

*The minimal time is ____ min.*

### Lesson

**Problem 12.2**

- *Write down the next three elements of the sequence:*

  *1, 2, 4, 8, 16, 32, ____ , ____ , ____ .*

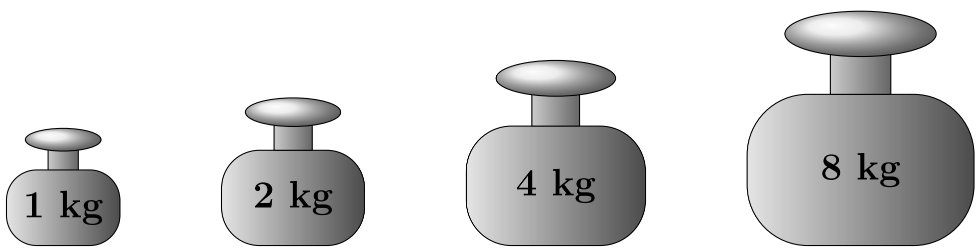
- *How do you get the next number from the previous one?*

The word *binary* means *relating to, composed of, or involving two things.* Sometimes, the word *binary* is shortened to a prefix *bi-*. For example, the word *bipedal* means *using only two legs for walking.* We humans are bipedal animals.

**Problem 12.3** *What other word with a prefix bi- do you know? What does the word mean?*

In this lesson, we will study the way of counting used in Binary Land, a country with arithmetic based on doubling the number one and the consecutive doubles.

In Binary Land, they only use the weights of 1, 2, 4, 8, etc. kilograms from the sequence above. They call them *basic weights.* They also call the numbers 1, 2, 4, 8, etc. *basic numbers.* Another Binary Land custom is to use each of the basic weights no more than once per weighing.




For example, the combination of weights

$$9 = 4 + 4 + 1$$

for measuring the weight of 9 kg is not allowed in Binary Land because it uses the 4 kg weight twice. The only allowed combination is

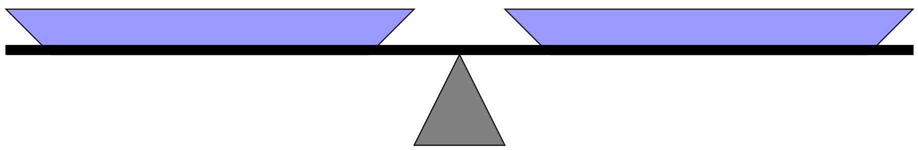
$$9 = 8 + 1.$$

The custom of using no more than one basic weight per weighing may seem strange for now, but you will realize that it is actually pretty smart soon enough.

**Question 12.1** *What is the difference between "less than" and "no more than"?*

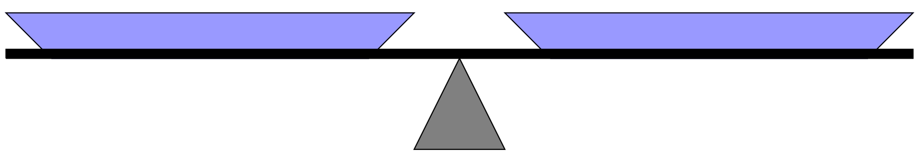
**Problem 12.4**

- *You put a 5 kg melon on the left plate of a balance scale. What basic weights do you put on the right plate to balance the melon? Draw the corresponding picture on the balance scale below. Then answer the question.*



*The basic weights are ____ and ____ kg.*

- *You put an 11 kg watermelon on the left plate of a balance scale. What basic weights do you put on the right plate to balance the watermelon? Draw the corresponding picture on the balance scale below. Then answer the question.*



*The basic weights are ____ , ____ , and ____ kg.*

Let us make a table that shows how they weigh objects in Binary Land. The top row of the table is a list of basic weights. The left-most number is the weight of the measured object, 5 kg in the first case. If a basic weight is used in the weighing, we put 1 underneath. If a basic weight is less than the measured weight, but not used in the weighing, we put 0. If a basic weight is greater than the measured weight, it cannot be used in the weighing. In this case, we leave the space under the weight blank.

Here is a table for the 5 and 11 kg objects from problem 12.4.

| measured weight | basic weights | | | | | | |
|---|---|---|---|---|---|---|---|
| | **64** | **32** | **16** | **8** | **4** | **2** | **1** |
| 5 | | | | | 1 | 0 | 1 |
| 11 | | | | 1 | 0 | 1 | 1 |

## Measuring 5 kg in Binary Land

• The following formula represents five as a sum of basic Binary Land numbers.

$$5 = 4 + 1$$

This way, the basic weights of 4 and 1 kg are used to balance the 5 kg weight on a balance scale. We write 1 underneath the weights.

• The basic weight of 2 kg is less than the measured weight of 5 kg, but it is not used to balance it on the scale. In the table, we write 0 underneath.

• The basic weights 8, 16, 32, and 64 kg are greater than the measured weight of 5 kg. We leave the space under the weights blank.

In the binary notations, 5 = B 101. We write B in front a number to show that it is written in the Binary Land language.

### Measuring 11 kg in Binary Land

• The following formula represents eleven as a sum of basic Binary Land numbers.

$$11 = 8 + 2 + 1$$

This way, the basic weights of 8, 2, and 1 kg are used to balance the 11 kg weight on a balance scale. We write 1 underneath the weights.

• The basic weight of 4 kg is less than the measured weight of 11 kg, but it is not used to balance it on the scale. In the table, we write 0 underneath.

• The basic weights 16, 32, and 64 kg are greater than the measured weight of 11 kg. We leave the space under the weights blank.

In the binary notations, $11 = B$ 1011.

**Problem 12.5** *Fill out the table below.*

| *measured weight* | *basic weights* | | | | | | |
|---|---|---|---|---|---|---|---|
| | **64** | **32** | **16** | **8** | **4** | **2** | **1** |
| 1 | | | | | | | |
| 2 | | | | | | | |
| 3 | | | | | | | |
| 4 | | | | | | | |
| 6 | | | | | | | |

**Problem 12.6** *Write the following numbers in the binary notations.*

$3 = B$ __________ $4 = B$ __________

$5 = B$ __________ $6 = B$ __________

$7 = B$ __________ $8 = B$ __________

It is harder to find binary notations for bigger numbers. Give the following a try. Don't feel upset if you cannot solve problem 12.7 right away. We will soon show you a general method that will help you solve similar problems with ease.

**Problem 12.7** *Represent the following numbers as sums of basic numbers. Then write them down in the binary notations.*

$50 =$ ____________ $100 =$ ____________

$50 = B$ __________ $100 = B$ __________

**Example 12.1** *To find the binary notation for the decimal number* 100*, let us start from* 1 *and keep doubling until we find the first basic Binary Land number greater than* 100*.*

$$1, \quad 2, \quad 4, \quad 8, \quad 16, \quad 32, \quad 64, \quad 128$$

*The last number,* 128*, is greater than* 100*. It cannot be a part of its binary decomposition. Let us take the previous number,* 64*, and subtract it from* 100*. We get* $100 - 64 = 36$ *or alternatively*

$$100 = 64 + 36.$$

*The smallest basic number less than or equal to* 36 *is* 32*. Note that* $36 - 32 = 4$*. The last number,* 4*, is itself basic.*

$$100 = 64 + 32 + 4 \tag{12.1}$$

*Formula (12.1) shows how to weigh a 100 kg object in Binary Land.*

| *measured weight* | *basic weights* | | | | | | |
|---|---|---|---|---|---|---|---|
| | **64** | **32** | **16** | **8** | **4** | **2** | **1** |
| 100 | 1 | 1 | 0 | 0 | 1 | 0 | 0 |

*Equivalently,* $100 = B\ 1100100$.

**Problem 12.8** *Use the abacus to check that* $64 + 32 + 4 = 100$.

Let us now describe the method used in example 12.1 step-by-step.

**Step 1**: find the greatest basic number less than or equal to the given number. Put 1 underneath the basic number in the table.

If the given number is 100, then the greatest basic number less than or equal to 100 is 64.

**Step 2**: Subtract the found basic number from the given number.

$$100 - 64 = 36$$

**Step 3**: Take the difference as the new given number. Return to step 1. Continue until the difference is zero.

The greatest basic number less than or equal to 36 is 32.

$$36 - 32 = 4$$

The greatest basic number less than or equal to 4 is 4.

$$4 - 4 = 0$$

The algorithm stops.

**Step 4**: Read off from the table and write down the binary notation for the original number.

$$100 = \text{B } 1100100$$

In the following problem, we use the four-step algorithm to find the binary notation for the number 184.

**Problem 12.9**

• *Find the greatest basic number less than or equal to* 184. *Use the abacus to double basic numbers if needed.*

*The basic number is* ______ .

• *Subtract the found basic number from* 183. *Use the abacus if needed.*

*The difference is* ______ .

• *Find the greatest basic number less than or equal to the difference you just found.*

*The next basic number is ______ .*

- *Subtract the found basic number from the difference.*

*The next difference is ______ .*

- *Find the greatest basic number less than or equal to the difference you just found.*

*The next basic number is ______ .*

- *Subtract the found basic number from the difference.*

*The next difference is ______ .*

- *Find the greatest basic number less than or equal to the difference you just found.*

*The next basic number is ______ .*

- *Subtract the found basic number from the difference.*

*The next difference is ______ .*

- *Is the last difference equal to zero? Circle the correct answer.*

$$Yes \qquad\qquad No$$

- *Fill the table below. Don't forget to put zeros under the basic weights you did not need.*

| *measured weight* | *basic weights* | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **128** | **64** | **32** | **16** | **8** | **4** | **2** | **1** |
| 184 | | | | | | | | |

- *Write down the binary notation for the number* 184.

$$184 = B \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ }$$

**Problem 12.10** *Use the four-step algorithm to find the binary notation for the number* 65.

| *measured weight* | *basic weights* | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **128** | **64** | **32** | **16** | **8** | **4** | **2** | **1** |
| 65 | | | | | | | | |

$$65 = B \quad \underline{\hspace{6cm}}$$

**Problem 12.11** *Use the four-step algorithm to find the binary notation for the number* 70.

| *measured weight* | *basic weights* | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **128** | **64** | **32** | **16** | **8** | **4** | **2** | **1** |
| 70 | | | | | | | | |

$$70 = B \quad \underline{\hspace{6cm}}$$

**Problem 12.12** *Would you need the basic weight of 2 kg to weigh a 60 kg in object Binary Land? Why or why not?*

**Example 12.2** *Finding the decimal notation for the binary number* $B\ 11011$ *is equivalent to replacing the question mark in the table below by a sum of the basic weights used in the weighing.*

| *measured weight* | *basic weights* | | | | | | |
|---|---|---|---|---|---|---|---|
| | **64** | **32** | **16** | **8** | **4** | **2** | **1** |
| ? | | | 1 | 1 | 0 | 1 | 1 |

$$B\ 11011 = 16 + 8 + 2 + 1 = 27$$

**Problem 12.13** *Find decimal notations for the following binary numbers. Use the abacus for adding basic binary numbers if needed.*

$$B\ 10 = \underline{\qquad\quad} \qquad\qquad B\ 11 = \underline{\qquad\quad}$$

$$B\ 101 = \underline{\qquad\quad} \qquad\qquad B\ 110 = \underline{\qquad\quad}$$

$$B\ 11111 = \underline{\qquad\quad} \qquad\qquad B\ 10000 = \underline{\qquad\quad}$$

**Problem 12.14** *Write down a number composed of the digits*

$$1,\ 1,\ 2,\ 2,\ 3,\ 3,\ 4,\ 4$$

*in such a way that there is one digit between the ones, two digits between the twos, three digits between the threes, and four digits between the fours.*

**Homework**

Finish all the problems unfinished in class.

**Problem 12.15** *Use the four-step algorithm to find the binary notation for the number* 22*.*

| *measured weight* | *basic weights* | | | | | | |
|---|---|---|---|---|---|---|---|
| | **64** | **32** | **16** | **8** | **4** | **2** | **1** |
| 22 | | | | | | | |

$$22 = B \quad \underline{\hspace{6cm}}$$

**Problem 12.16** *Use the four-step algorithm to find the binary notation for the number* 99.

| *measured weight* | *basic weights* | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **128** | **64** | **32** | **16** | **8** | **4** | **2** | **1** |
| 99 | | | | | | | | |

$$99 = B \quad \underline{\hspace{6cm}}$$

**Problem 12.17** *Fill out the table below.*

| *measured weight* | *basic weights* | | | | | | |
|---|---|---|---|---|---|---|---|
| | **64** | **32** | **16** | **8** | **4** | **2** | **1** |
| 7 | | | | | | | |
| 8 | | | | | | | |
| 9 | | | | | | | |
| 10 | | | | | | | |
| 12 | | | | | | | |
| 13 | | | | | | | |
| 14 | | | | | | | |
| 15 | | | | | | | |

**Problem 12.18** *Find decimal notations for the following binary numbers. Use the abacus for adding basic binary numbers if needed.*

$B\ 111 =$ ______ $B\ 1000 =$ ______

$B\ 1010 =$ ______ $B\ 1110 =$ ______

$B\ 101011 =$ ______ $B\ 1011011 =$ ______