

A π day celebration!

Everyone's favorite geometric constant!

Math Circle

March 10, 2019

The circumference of a circle is another word for its perimeter. A circle's circumference is proportional to its radius.

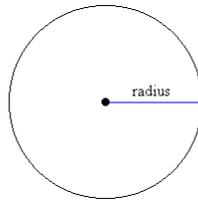


Figure 1: A circle of radius r .

1. Please answer the following:

- (a) For some radius r , the circumference of a circle of radius r is 20 feet. What is the circumference of a circle when the radius is doubled?

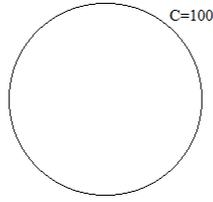


Figure 2: A circle with circumference 100.

- (b) The circle in Figure 2 has a circumference of 100. How many times bigger is the radius of a circle whose circumference is 300?

2. A mathematician noticed that the ratio between the circle's circumference and its diameter is constant, so that the circumference is roughly equal to three times the diameter, or six times the radius. The Greek letter π , the first letter for the Greek word for "perimeter," is used to denote this mathematical constant.

$$\pi = \frac{\text{Circumference}}{\text{Diameter}}$$

$$\text{Circumference} = \pi \cdot \text{Diameter}$$

$$\text{Circumference} = 2\pi \cdot \text{Radius}$$

Solve for the following:

- (a) Radius = 3, Circumference =

- (b) Radius = 5, Circumference =

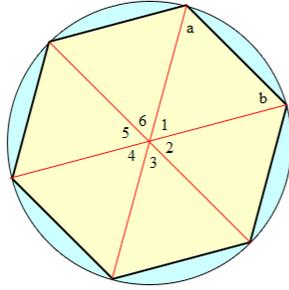


Figure 3: A regular hexagon inscribed within a circle.

(c) Circumference = 6π , Radius =? Diameter =?

(d) Circumference = 10, Radius =? Diameter =?

3. Figure 3 depicts a regular hexagon inscribed in a circle with radius r . Consider the following. Fill in the details as necessary.

(a) A full circle is 360 degrees, therefore each of the angles labeled in 1 - 6 is:

(b) Each triangle in Figure 3 is isosceles because:

(c) Angles a and b are equal because:

(d) Angle $a =$

(e) Triangles 1 - 6 are all isosceles, but more precisely they are all:

(f) The length of the arc of the circle connecting a to b is longer than the side of the hexagon connecting a to b because:

(g) What is the circumference of the circle in Figure 3? What about the perimeter of the hexagon?

(h) The above argument shows that π is larger than some number. What number is that?

4. It is hard to know exactly, but we think that cavemen approximated $\pi \approx 3$, and ancient Greeks approximated $\pi \approx 22/7$.

(a) A caveman sees a circle of radius 7 inches. How large would they predict the perimeter to be?

(b) Same question as above, but for an ancient Greek.

(c) What is the absolute difference between these two calculations?

(d) For a circle of any radius (not just 7), a caveman and an ancient Greek would always differ by a constant ratio. What is this ratio?

- (e) A caveman measured the radius of a circle and computed the circumference to be 84 inches. What circumference would an ancient Greek have calculated? Solve this problem in two ways, one by calculating the true radius, and the other by using your answer to the last question.

5. Please calculate the perimeter of the following in terms of π .

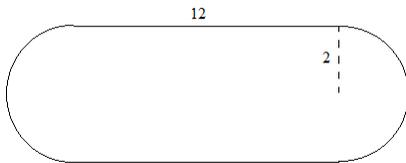


Figure 4: (a)

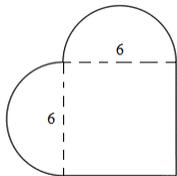


Figure 5: (b)

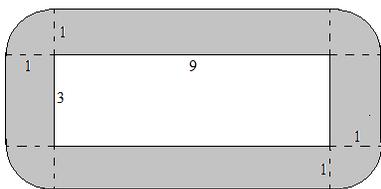


Figure 6: (c)

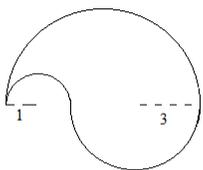


Figure 7: (d)

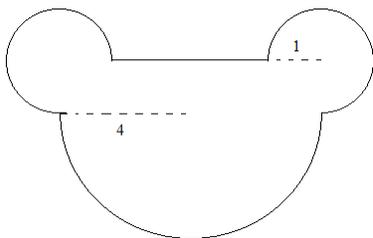


Figure 8: (e)

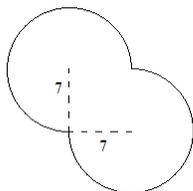


Figure 9: (f)

6. Now let's talk about how you can (almost) derive the formula for the area of a circle.

- (a) One way to estimate the area of a circle is to cut the circle into a pizza shape, like Figure 11. If you cut a circle of radius r into $2n$ pieces, what is the length of the two similar sides of each pizza slices? Using this, approximate the height of the rectangular shape in the pizza rearrangement.

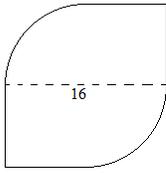


Figure 10: (g)

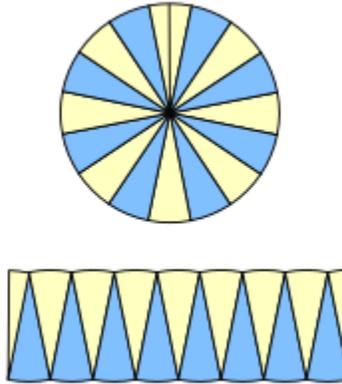


Figure 11: Cutting the circle into a pizza and rearranging to form a (near) rectangle. Image taken from https://en.wikipedia.org/wiki/Area_of_a_circle

Check with you assistant that you got the correct answer.

- (b) How can you relate the length of the top and bottom of the rectangular shape to the circumference of the original circle? Using this, what is the approximate width of the rectangular shape? Check with you assistant to make sure that you got the correct answer.

- (c) As you cut the circle into smaller and smaller pieces, the lower figure starts to look more and more like a true rectangle. You can prove (using calculus!) that as you make the number of pieces very large, this shape becomes exactly a rectangle. Based on your answer to the previous two questions, what is the area of this rectangle? What does this say about the area of the original circle?

7. Now let's do some more calculations like question 5, This time let's calculate the area of each figure!

- (a) Calculate the area of Figure 4.

- (b) Calculate the area of Figure 5.

(c) Calculate the area of Figure 6.

(d) Calculate the area of Figure 7.

(e) Calculate the area of Figure 8.

(f) Calculate the area of Figure 9.

(g) Calculate the area of Figure 10.

8. Please do the following miscellaneous π related problems:

(a) A vine starts at the bottom of a 28 foot tree trunk, and wraps around to the top going around exactly 7 times. If the diameter of the trunk is 3 feet, what is the length of the vine?

- (b) For the moment, pretend that the Earth is a perfect sphere, with no hills, valleys, oceans, etc... A belt is wrapped around the equator of the earth and pulled tight. Then, this belt is loosened by 10 feet, so that the equator has some slack. If the belt is held at a uniform distance above the ground, how much room would there be between the Earth's surface and the loosened belt? Make an educated guess and then confirm your answer. The circumference of the Earth is about 131,000 feet.
- (c) As far as I know, the radius of the known universe is about $4.5 \cdot 10^{26}$ meters. Imagine that you had to measure the circumference of a circle around the edge of the known universe. Because π isn't known exactly, you have to approximate it. If you use $\pi \approx 3.14$ you'll make an error in your measurement. If you use $\pi \approx 3.14159$ you'll still make an error, but a smaller one. Say that you want to make an error of 1 meter or less. How many digits of π should you use? Make a guess and then compute an answer. You don't need an exact answer, but I want an estimate within one or two.

- (d) The smallest possible length of *anything* in the universe is called a Planck length (named after the physicist Max Planck), which is about $2 \cdot 10^{-35}$ meters. Can you also answer the above question if you want to make an error less than one Planck length?

9. Challenge problem time! Have you ever heard someone say that π is irrational and so it contains every whole number in its decimal expansion? Let's take a closer look at that...

- (a) I won't ask you to prove it, but π is indeed irrational. Prove that this means that its decimal digits must go on forever.

- (b) What are some other numbers whose digits go on forever? This of at least two that are rational, and two that are irrational.

- (c) People are fond of saying that the digits of π are "random," but the digits of π are definitely not random. They can only be one thing, the digits of π ! What do you think that people mean when they say this? What do they mean by "random?" Check with your lead instructor once you have an answer.

- (d) Consider the following number called the Champernowne constant:

0.123456789101112113...

Does this number contain every number in its decimal expansion? Do you think that most people would say that this number is random (in the way that they mean for π)?

- (e) Michael defined a new number called The Puthawala constant. It is defined as the following. Write down the digits of π and replace every even digit with 0, and odd digit with 1. The Puthawala constant is also irrational. Does it contain every possible number in its decimal expansion? It is 'random?'

- (f) Do you think that the digits of π contain every possible whole number? Believe it or not, no one actually knows for sure! Try and change this by proving that the digits are, or are not random. Whichever you thought.