Combinatorics

Count your blessings

without replacement

Los Angeles Math Circle

March 3rd, 2019

Before we move onto to the new stuff, let's do a couple of warm-up problems.

1. Answer all of the following please!

   (a) How many ways can you arrange the letters of the word "EASY"?

   (b) Michael came up with a new language called Puthawa that uses 7 distinct letters (a, b, c, d, e, f, g). You can form a word in Puthawa by combining letters to form four letter words. Your word can not be the same letter four times in a row. How many words does Puthawa have?

(c) Cassandra and Christine came up with their own language Tai (not to be confused with Thai) too. Tai has 22 "normal" letters, and an additional 4 "final" letters which can only be used as the last letter of a word. Words in Tai can end either with a normal or a final letter. If this language contains only words with three, four, or five letters. How many different words can be written in this language?

(d) There is a board game called "Settlers of Catan," in which players colonize a 19 piece hexagonal board. This board is made up of

- 4 Sheep tiles
- 4 Wood tiles
- 4 Wheat tiles
- 3 Ore tiles
- 3 Brick tiles
- 1 Desert tile

If each tile of the same kind is identical, then how many unique Settlers of Catan boards are there?

2. Now I want to spend a bit of time making sure that you are comfortable with the factorial notation. Throughout all of these questions, n represents an unspecified natural number.

(a) Simplify as one factorial:

$$5! \cdot 6 \cdot 7 =$$

(b) Compute the following:

$$\frac{27!}{25!} =$$

$$\frac{n!}{(n-3)!} =$$

$$\frac{(2n-1)!}{(2n-3)!} =$$

(c) Which quantity is larger, $\frac{(n+1)!}{(n-1)!} - \frac{n!}{(n-1)!}$ or n^2?

(d) What is the difference between $(n^2)!$ and $(n!)^2$ for any natural number n? Which one do you think is larger? Make a guess and then check in the case that $n = 3$.

(e) Please find all natural numbers k for which the following is true:

$$\frac{(2k)!}{(2k-1)!} = \frac{(k+6)!}{(k+5)!}$$

3. Let's do a little bit of combinatorics on a chess board. [1] A chess board is 8×8, and is checkerboarded so that half of the squares are white, and the other half are black.

[1] Problems in this section have been taken from N. Ya. Vilenkins "Combinatorics."

(a) In how many ways can we choose a black square and a white square on a chessboard if the two squares must not belong to the same row or column?

(b) A chessboard is 8×8. A game of chess is played with two sets of pieces in white and black. Each set contains 8 pawns, 2 rooks, 2 knights, 2 bishops, 1 queen and 1 king. How many ways can all of the pieces be arranged on the chess board?

(c) Two rooks can attack each other if they are on the same row or column. How many ways can you place six rooks on the chess board so that none of them can attack each other?

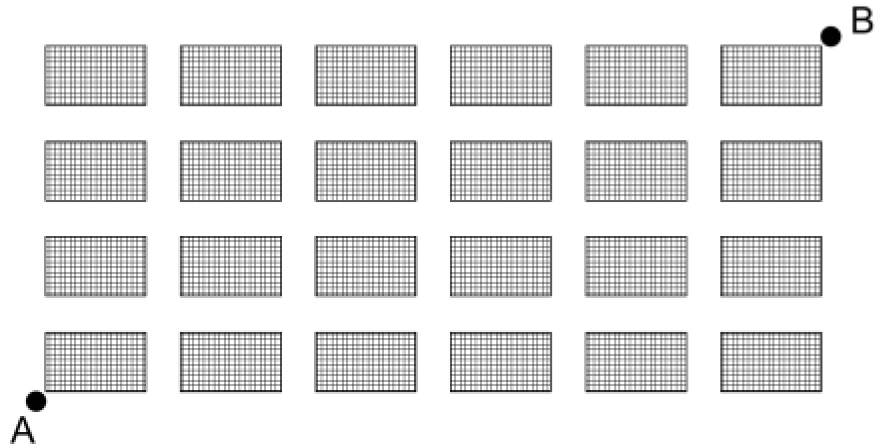

Figure 1: New York (Arial View)

4. Let's do a few more problems involving the number of ways that all have to do with finding paths.

   (a) You are visiting a part of New York City that is laid out like a grid (see the Figure 1). You start at the block corner marked A, and want to get to block corner marked B. Point A is 6 blocks West of B, and 4 blocks south. If you want can only walk north and east, how many ways can you get from A to B?

   (b) Can you answer the same question if point B is m blocks East of A and n blocks north? Give you answer using factorial notation.

(c) Let A be at the point $(0, 0)$ on the plane, point B be at the point (m, n) and C be at the point (o, p) where $o \geq m$ and $p \geq n$. How many ways can you get from A to B and then to C if you can only move either up or to the right?

5. **These last couple problems are challenge problems, and considerably more difficult than the ones that came before them. Solve them if you dare, you've been warned!**

   (a) Let's suppose that you want to move form the point $(0, 0)$ to the point (n, n) in the same way as before but with a special catch. You don't ever want to move to a point (x, y) such that $y > x$. So, for example, you can't move to the point $(1, 2)$ or $(3, 4)$ but $(2, 1)$ or $(3, 3)$ is Ok. If $n = 4$, how many paths can you take from A to B?

   (b) Same question, except what if $n = 5$?

(c) Consider but **do not simplify** the following mathematical expressions:

$$((2+4)*4)^2$$
$$((1+3)-(4/7))^2$$
$$(4)^3 - (6)^-1 * ((((3-4)^2) * (9-2)) * 7)^-1.$$

If you erase all of the operators and numbers and just look at the parenthesis, then you are left with:

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We say that the above parenthesis strings are all balanced, because can all be obtained by writing down an expression and erasing numbers and operations. Not all parenthesis are strings are balanced, for example:

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are all invalid. A sequence S of (and)'s is balanced under two conditions. The first condition is that there are the same number of ('s and)'s. What is the second condition?

(d) Before starting this question, check to make sure with an instructor to make sure that you got the last one right. How many valid strings of 8 and 10 parenthesis are there?