Lesson 7: Invariants and the Intercept Theorem

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1 From Last Time

Problem 1.
a) Consider an $n \times m$ table filled with integers. With one operation, you are allowed to take any row or column and and negate every number in that row/column. Show that it is possible to make sure every row and column has nonnegative sum using such operations.

b) Same problem with real numbers in the table, not integers.

Problem 2.
Consider n segments on the plane with $2n$ distinct endpoints. The following process is performed: if two segments AB and CD intersect, we replace them by segments AD and BC. Show that eventually no two segments will intersect.

2 New Problems

Problem 1 (Intercept Theorem)**.**
a) Consider two rays r, ℓ out of point O and distinct points $A_1, \dots, A_n$ on r such that

$$A_1A_2 = A_2A_3 = \ldots = A_{n-1}A_n$$

Show that if $B_1, \dots, B_n$ are points on ℓ such that the lines A_iB_i are parallel to each other for all $1 \le i \le n$, then

$$B_1B_2 = B_2B_3 = \ldots = B_{n-1}B_n$$

Hint: We proved the case $n = 2$ last week in problem L6.5. For the general case, use the statement for $n = 2$ to show that $B_1B_2 = B_2B_3$, then use it to show that $B_2B_3 = B_3B_4$, and so on to conclude the problem.

b) With O, r, ℓ as in part a), let A, B, C be points on r such that AB/BC is an integer. Show that if A', B', C' are points on ℓ such that $AA' \parallel BB' \parallel CC'$, then $A'B'/B'C' = AB/BC$.
Hint: let $AB/BC = n$. Set up points $A_1, \dots, A_{n-1}$ on AB such that

$$AA_1 = A_1A_2 = \ldots = A_{n-1}B = BC$$

then use part a).

c) Same as part b), except AB/BC is rational.
Hint: if $AB/BC = m/n$, add some extra points on both the segment AB and the segment BC similarly to part b), then use 1a).

Problem 2.
Let AB be a given segment, and n be a positive integer. Use straightedge and compass to split AB into n equal parts. You may assume the result of problem 1a).
Hint: Construct a random auxiliary line ℓ through A and points $A_0, \ldots, A_n$ on ℓ such that $A_0 = A$ and

$$A_0A_1 = A_1A_2 = \ldots = A_{n-1}A_n$$

Now use 1a).

Problem 3.
Let $ABCD$ be an arbitrary quadrilateral. If M, N, P, Q are the midpoints of AB, BC, CD, DA respectively, show that $MNPQ$ is a parallelogram.