Combinatorics (Part II)

BEGINNERS 02/08/2015

Warm-Up

(a) How many five-digit numbers are there?

$\underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }$
$9 \times 10 \times 10 \times 10 \times 10 = 90000$

or 99999 – highest 5-digit
-9999 – highest 4-digit
90000

(b) How many are odd?

$\underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }$
$9 \times 10 \times 10 \times 10 \times 5$
$= 45,000$

or $\frac{90,000}{2} = 45,000$

(c) How many are odd and larger than 30,000?

$\underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }$
$7 \times 10 \times 10 \times 10 \times 5 = 35,000$ or $\frac{70,000}{2} = 35,000$

(d) How many have only odd digits?

$\underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }$
$5 \times 5 \times 5 \times 5 \times 5 = 5^5$

(e) How many have only even digits?

$\underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }$
$4 \times 5 \times 5 \times 5 \times 5 = 4 \times 5^4$

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Part IV: Permutations (Continued)

Permutations are arrangements that can be made by placing objects in a row. The order of the objects is important.

1. How many three digit numbers can you write using the digits 3, 3 and 4?

(a) Let's say that we have a 3, a **bold 3** and a 4. Write down all the numbers that you can make with the digits 3, **3** and 4.

334 34**3**

3**3**4 4**3**3

343 43**3**

(b) Considering the 3 and the bold **3** as two different digits, how many different numbers are there? 6

Even though they are colored differently, the 3 and the bold **3** have the same meaning:

3**3**4 is the same number as **3**34.
4**3**3 is the same number as 43**3**.
343 is the same number as 34**3**.

For every permutation that we write, there will be another one in which the positions of the 3s are switched. And both numbers will obviously be the same!

Therefore, there are some *repetitions* among the numbers above.

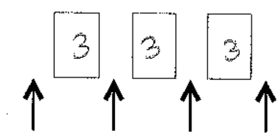
(c) Taking into account that 3 and bold **3** mean the same digit, how many different numbers written with digits 3, 3 and 4 are there?

3

Here is how we computed the number of arrangements of the digits 3, 3, 4:
First, pretend that the two 3s are different (one of them is bold). Then the number of arrangements is equal to 3! = 6. Then, take into account that the arrangements which are obtained from each other by switching the two 3s are actually the same. As a result, we get 3! / 2! = 3.

2. How many four-digit numbers can you write using the digits 3, 3, 3, and 4?

   (a) Notice that three of the digits are equal to 3. This means that we just have to decide where to put the digit 4.

   This can be done in ___4___ ways.

   Thus, the total number of permutations is ___4___.

   (b) Now we will compute the number of rearrangements using the same method as in the first problem. Let's say that we have a 3, a bold **3**, an underlined 3, and a 4. In how many ways can you rearrange 3, **3**, 3, and 4?

   $4! = 24$

   (c) Every two rearrangements which differ just by the order of 3s are the same. In how many ways can you rearrange 3, **3**, 3 in three slots?

   $3!$

   (d) How many distinct four-digit numbers written with 3, 3, 3, and 4 are there? Does your answer agree with the result of part (a)?

   $\frac{4!}{3!} = 4$

   Yes.

Word Bank

1. In how many ways can you rearrange the letters of the word IOWA?

   $4! = 24$

2. In how many ways can you rearrange the letters of the word ARIZONA?

   (a) Pretending that the two letters A are different, how many permutations do you have?

   $7!$

   (b) Now taking into account that both As are identical, how many permutations do you have?

   $\frac{7!}{2!}$

3. In how many ways can you rearrange the letters of the word COLORADO?

   $\frac{8!}{3!}$ You can leave the answer as i's.

4. In how many ways can you rearrange the letters of word TENNESSEE?

   $\frac{9!}{2!2!4!}$ Total no. of characters: 9
   Repetitions: 2 Ns, 2 Ss, 4 Es

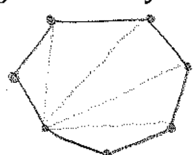
Practice Problems

1. How many diagonals are there in a quadrilateral?

2

2. A *diagonal* of a polygon is a line segment that connects any two non-adjacent vertices with each other. How many diagonals does a 7-sided polygon have?

(a) How many diagonals can you draw from one vertex?



4

(b) What if you do this for all seven vertices?

4 diagonals from each vertex.

4×7

You can always draw diagonals from a vertex to all vertices, except to the adjacent vertices and that vertex.

(c) Notice that in part (b) you counted each diagonal twice. How many diagonals does a 7-sided polygon have?

$$\frac{4 \times 7}{2} = 14$$

3. How many diagonals does a 100-sided polygon have?

$$\frac{100 \times 97}{2} = 97 \times 50 = 4850$$

4. How many diagonals does an n-sided polygon have?

$$\frac{n(n-3)}{2}$$

Practice Problems

5. There are 39 cities in a country, each with one airport. Every pair of them is connected by an air route. How many air routes are there?

$$\frac{39 \times 38}{2}$$

6. Josh's mother has two apples, three bananas, and four oranges. Every morning, for nine days, she gives one fruit to her son for breakfast. How many ways are there to do this?

You can think of this problem as arranging the 9 letters A, A, B, B, B, O, O, O, O in a row.

$$\frac{9!}{2!\,3!\,4!}$$

7. How many "words" can be written using exactly five As and at most three Bs?

Exactly 5 As and no B : $\frac{5!}{5!} = 1$

and 1 B : $\frac{6!}{5!} = 6$

and 2 Bs : $\frac{7!}{5!\,2!}$

and 3 Bs : $\frac{8!}{5!\,3!}$

So, total no of words =

$$1 + 6 + \frac{7!}{5! \times 2!} + \frac{8!}{3!\,5!}$$

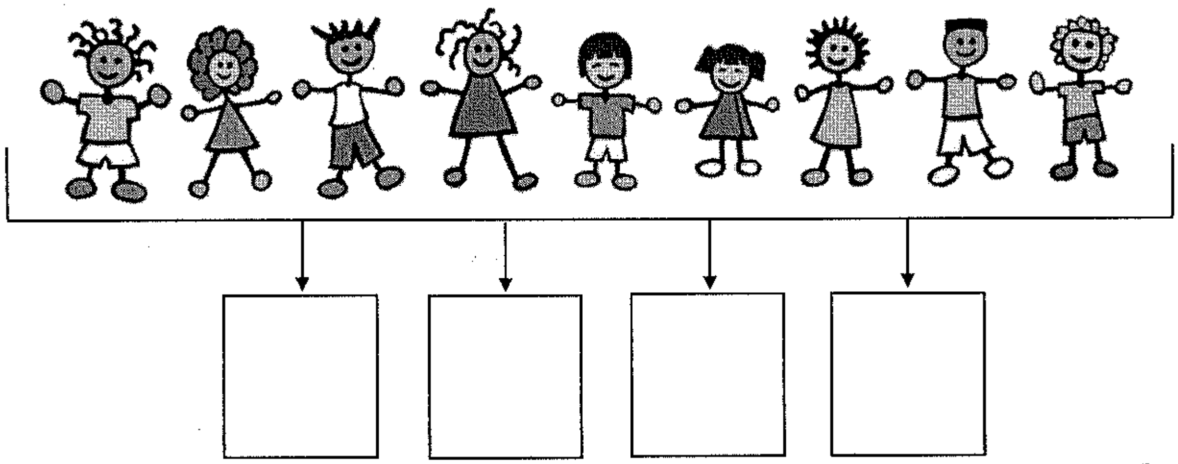
$$7 + 21 + 56$$

$$= 84$$

Part V: Combinations

Sometimes, the order in which you arrange things doesn't matter. For example, Ms. Cranberry has to choose four students from her class of nine to send for a mathematical contest. Does the order in which she picks the first, second, third and fourth student matter?

No, it does not!



i. How many different options does Mrs. Cranberry have for the first student? 9

ii. After selecting the first, how many options does she have for the second? 8

iii. Now, for the third? 7

iv. For the fourth? 6

We see that there are 9 × 8 × 7 × 6 ways of choosing the students. Let us call this number A. However, this gives us the answer for the number of permutations (i.e., the order matters!)

Forget about the order now. If the names of the four students picked are Abe, Gus, Rob and Zed, it doesn't matter if the order is "Gus, Zed, Abe, Rob" or "Zed, Abe, Rob, Gus." So, in how many ways can you arrange these four students amongst themselves? 4 × 3 × 2 × 1

Let us call this number B. B is simply the number of ways in which you can rearrange four students. It is also the number of repetitions, right?

Similarly to what we did above when we had repetitions or *redundancies*, let's divide A by B. We find the number of all the permutations and divide by the number of all the *redundancies*.

What does $\frac{A}{B}$ show? Explain in your own words.

It shows the total no. of ways of choosing 4 children from a group of 9. We divide by 4! because the order does not matter.

Practice Problems

1. There are four kittens at the pet store. Your mom says that you may choose two to take home. In how many different ways can you choose two kittens from the litter of four?

$$\frac{4 \times 3}{2!} = 6$$

2. The coach of the UCLA football team has to choose a captain and a deputy from his team of 10 students. How many ways are there to do that?

$10 \times 9 = 90$ (The order matters because captain and deputy are not equivalent)

3. The coach of the USC football team, however, wants to choose a captain and a deputy as well as three assistants from his team of 10 students. In how many ways can he pick these four leaders?

$10 \times 9 \times \frac{8 \times 7 \times 6}{3!}$ (because the order in which the assistants are picked does not matter)

4. Mr. Pi has to choose three girls and three boys to send for a debate. There are 14 girls and 11 boys in his class. In how many ways can he make the team?

$$\frac{14 \times 13 \times 12}{3!} \times \frac{11 \times 10 \times 9}{3!}$$

5. On any given night, there can be between zero and four babysitters at home in the Simpsons' house. The babysitter company has eight employees from which it can choose to send babysitters to the Simpsons. How many possible combinations of babysitters can be at the house?

zero babysitters = 1 way

1 babysitter = 8 ways

2 babysitters = $\frac{8 \times 7}{2!}$

3 babysitters = $\frac{8 \times 7 \times 6}{3!}$

4 babysitters = $\frac{8 \times 7 \times 6 \times 5}{4!}$

Total no. of ways:

$1 + 8 + \frac{8 \times 7}{2!} + \frac{8 \times 7 \times 6}{3!} + \frac{8 \times 7 \times 6 \times 5}{4!}$

$= 163$

Challenge Yourself!

1. Grace's mother has four fruits: an apple, an orange, a pear and a banana. In how many ways can she pack a snack for Grace if she can give her either one, two, three or four fruits?

1 fruit: 4 ways = 4

2 fruits: $\frac{4 \times 3}{2!}$ ways = 6

3 fruits: $\frac{4 \times 3 \times 2}{3!}$ ways = 4

4 fruits: $\frac{4 \times 3 \times 2 \times 1}{4!}$ ways = 1

Total no. of ways

4 + 6 + 4 + 1

= 15 ways

2. (Math Kangaroo) A pizza parlor sells small, medium, and large pizzas. Each pizza is made with cheese, tomatoes, and at least one of the following toppings: mushroom, onion, peppers and olives. How many different pizzas are possible?

1 topping: 4 options

2 toppings: $\frac{4 \times 3}{2!}$ options = 6

3 toppings: $\frac{4 \times 3 \times 2}{3!}$ = 4 options

4 toppings: $\frac{4 \times 3 \times 2 \times 1}{4!}$ = 1 option

So, in one size, there are 4 + 6 + 4 + 1 = 15 options.

So, 15 × 3 in all 3 sizes.

So, 45 different pizzas are possible.

3. (Math Kangaroo) The body of a certain caterpillar is made up of five spherical parts, 3 of which are yellow and 2 are green. What is the greatest possible number of different types of caterpillar that could exist?

$$\frac{5!}{2!\,3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1}$$

= 10 types of caterpillars

Challenge Yourself!

4. Do seven-digit numbers with no digits 1 in their decimal representations constitute at least 50% of all seven-digit numbers?

(a) First, determine the numbers of all seven-digit numbers?

$$9 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$$

$$9 \times 10^6$$

(b) Then, determine the numbers of all seven-digit numbers with no digits 1.

$$8 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9$$

$$8 \times 9^6$$

(c) Does your answer in part (b) constitute at least 50% of your answer in part (b)?

$$\frac{9^6 \times 8}{9 \times 10^6} = \frac{9^5 \times 8}{10^6} = (.8) \times (.9)^5$$

This is less than 0.5.

No.

5. A teacher has made ten statements for a True/False test.
(a) How many different answer sheets can be turned in?

2^{10}

(b) If there are four true statements and six false statements, how many distinct answer keys could there be for the test?

$\frac{10!}{4!\,6!}$ → you are picking any four statements to be true and assigning the rest to be false.

Challenge Yourself!

6. We toss a dice 3 times. Among all the possible outcomes, how many have at least one occurrence of six?
(a) First, determine the number of total possible outcomes for three dice tosses?

$6 \times 6 \times 6$
$= 216$ total possible outcomes

(b) Determine the number of total possible outcomes with no occurrence of a six?

$\overline{5} \times \overline{5} \times \overline{5}$ 125 ways

(c) Finally, find the number of outcomes with at least one occurrence of six?

$216 - 125$ ways with at least one six.
$= 91$ ways

7. How many ways are there to put one white and one black rook on a chessboard so that they do not attack each other?

[You cannot place the second rook in the same row or column as the first one.]

64 ways to put a white rook.

Now, you can put the black rook on any of 49 spots that the white rook cannot attack

so, 64×49

8. How many ways are there to put eight rooks on a chessboard so that they do not attack each other?

Ways to put first rook: $8 \times 8 = 64$ ways (8 columns x 8 rows)
second rook: $7 \times 7 = 49$ ways (we cannot use the same row or column)
third rook: $6 \times 6 = 36$ ways
⋮
last rook: $1 \times 1 = 1$ way

Total no. of ways: $\frac{8^2 \times 7^2 \times 6^2 \times 5^2 \times 4^2 \times 3^2 \times 2^2 \times 1^2}{8!} = \frac{(8!)^2}{8!} = 8!$

(because the rooks are identical)

Challenge Yourself!

9. How many four-digit numbers have two even and two odd digits?
(a) If you start with an odd digit, in how many ways can you arrange the odd digits and even digits? [Hint: Consider listing all the possibilities: OOEE, ...]

OOEE
OEOE
OEEO

3 ways

(b) Determine the number of four-digit numbers that have two even and two odd digits and start with an odd digit. [Hint: Don't forget to account for the number of different possible orderings of even and odd digits.]

$$5 \times 5 \times 5 \times 5$$
$$= 5^4$$

So, 3×5^4 ways

(c) If you start with an even digit, in how many ways can you arrange the odd digits and even digits? [Hint: Again, consider listing the possibilities: EEOO, ...]

EEOO
EOEO
EOOE

3 ways

(d) Determine the number of four-digit numbers that have two even and two odd digits and start with an even number?

$4 \times 5 \times 5 \times 5 = 4 \times 5^3$ ways

(cannot start with 0)

So, $3 \times 4 \times 5^3$ ways

(e) Finally, find the number of four-digit numbers that have two even and two odd digits by adding up the results in parts (b) and (d):

$3 \times 4 \times 5^3 + 3 \times 5^4$ ways

$= 5^3(12 + 15)$

$= 27 \cdot 5^3$

Challenge Yourself!

10. In how many ways can you rearrange the letters of Mary Poppins' favorite "SUPERCALIFRAGILISTICEXPIALIDOCIOUS"?

$$\frac{34!}{3!\,3!\,2!\,2!\,2!\,2!\,3!\,3!\,7!\,2!}$$

$$= \frac{34!}{(3!)^4\,(2!)^5\,7!}$$