Lesson 6: Invariants and Geometric Constructions

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1 From Last Time

Problem 4.
A group of children is standing in a circle, and each child has an even number of candies. Every minute all children simultaneously give half of their candies to their neighbor on the right. If after such operation someone has an odd number of candies, they get one extra candy from the teacher. Show that at some point all children will have the same number of candies.

2 New Problems

Problem 1.
a) Consider an $n \times m$ table filled with integers. With one operation, you are allowed to take any row or column and and negate every number in that row/column. Show that it is possible to make sure every row and column has nonnegative sum using such operations.

b) Same problem with real numbers in the table, not integers.

Problem 2.
Consider n segments on the plane with $2n$ distinct endpoints. The following process is performed: if two segments AB and CD intersect, we replace them by segments AD and BC. Show that eventually no two segments will intersect.

Problem 3.
On a field in the shape of a 10×10 grid 9 squares are infested with weeds. A new square can get infested with weeds if at least two of its adjacent squares are infested. Two squares are called adjacent if they share a side. Show that there will always be a square on the field not infested with weeds. *Hint: Consider the perimeter of the shape infested with weeds.*

Problem 4.
Given three segments s_1, s_2, s_3 on the plane, construct a parallelogram with one of the sides equal to s_1 and the diagonals equal to s_2 and s_3. You may assume such a parallelogram exists.

Problem 5.
Consider two rays r, ℓ out of point O, a segment AB on r and point C on ℓ. Let M be the midpoint of AB, let D be the intersection of ℓ and the line through M parallel to AC and let E be the intersection of ℓ and the line through B parallel to AC. Show that D is the midpoint of CE.