

Hard Problems

Though problems are *really* difficult, they become a lot more manageable if you study them from the modular perspective (which is the way it should be, since we've spent the whole afternoon learning about modular arithmetic!) The trick will often be to pick the right number(s), to "mod out by".

Problem 1) How many zeroes are at the end of $100!$?

Problem 2) The equation $x^2 + y^2 + z^2 = 10xyz$ has no integral solutions, except $x = y = z = 0$. Prove it.

Problem 3) Given three integers, $\{a, b, c\}$, prove that we can find two of them – say, a and b – such that $10 \mid a^3b - ab^3$.

Problem 4) Define the sequence $\{a_n\}$ by setting $a_1 = 2, a_2 = 5$, and

$$a_{n+1} = (2 - n^2)a_n + (2 + n^2)a_{n-1}, \quad n \geq 2.$$

Do there exist indices, p, q, r such that $a_p \times a_q = a_r$?

Problem 5) 2^{29} is a nine-digit number, with no repeated digits. Which digit does not appear?

Problem 6) The last four digits of a perfect square are equal. Prove that they are equal to zero.

Problem 7) Given a prime p , and two integers, a, b , working mod p , express

$$(a + b)^p$$

in terms of a and b . (This result is coolest if Fermat's Little Theorem is taken into account.)

Problem 8) Earlier, we saw that for every prime p , $(p - 1)! \not\equiv 0 \pmod{p}$. Compute $1! \pmod{2}$, $2! \pmod{3}$, $4! \pmod{5}$, and $6! \pmod{7}$. Is there a pattern that emerges? Can you prove it?