LAMC Beginners' Circle **January 19, 2014**

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Warm-up

Problem 1 *Simplify the following fraction.*

$$\frac{\frac{a}{b} - \frac{c}{d}}{\frac{a}{d} + \frac{c}{b}} =$$

The following problem was communicated to me by one of our students, Arul Kolla.

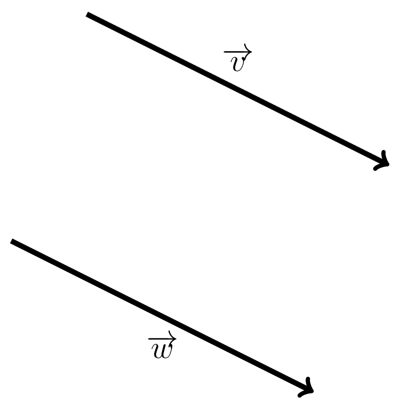
Problem 2 *Make the number 120 using nothing else but five zeroes and some symbols of math operations.*

A variation of the following problem was communicated to me by Mark Krone, the father of one of our students, Stella Krone.

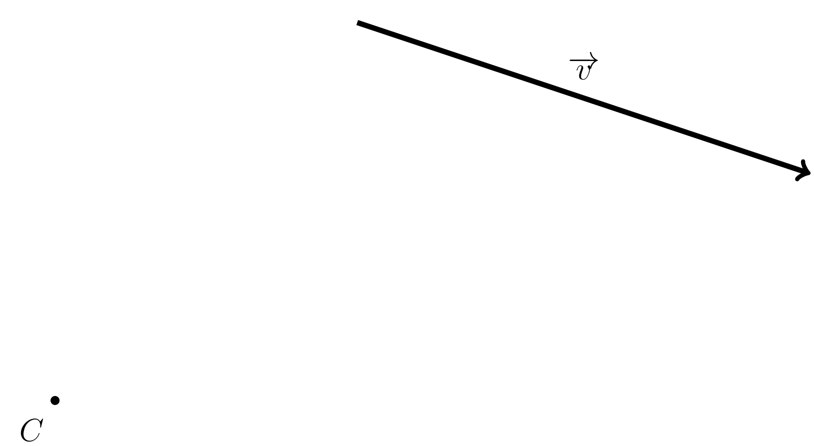
Problem 3 *In a dark room, there is a deck of 52 cards on the table. Ten of the cards face up, the rest of the deck face down. The facing up cards are randomly spread throughout the deck. Is it possible to split the deck into two parts, possibly having different size, so that the number of the cards facing up is the same for each of the parts, all without turning on the light?*

Back to vectors

Problem 4 *Is it possible to check whether $\overrightarrow{v} = \overrightarrow{w}$ on the picture below using only a compass? Why or why not?*

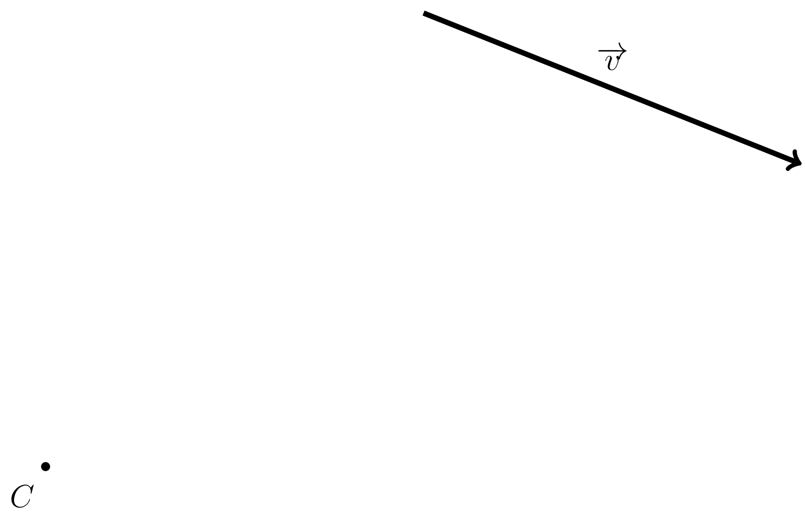


Problem 5 *Use a compass and a ruler to construct the vector* $\overrightarrow{w} = -.75\overrightarrow{v}$ *for the vector* $\overrightarrow{v}$ *given below such that point* C *is its terminal point.*



Note 1 *We know how to divide a segment into any positive integral number of parts. We also know how to construct a vector opposite to a given one. Combining these two procedures together, we can multiply a vector by any rational number, using a compass and ruler as tools. Multiplying a vector by a number that is not rational is a bit more tricky, but still quite doable.*

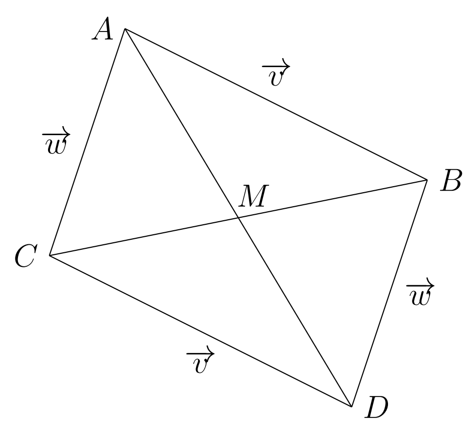
Problem 6 *Use a compass and a ruler to construct the vector* $\overrightarrow{w} = \sqrt{3}\overrightarrow{v}$ *for the vector* $\overrightarrow{v}$ *given below such that point* C *is its initial point. Hint: the Pythagoras' Theorem will help.*



Vectors are a very powerful tool. Below we will use vector algebra to re-prove some of the statements we have proven in the past.

Example 1 *Prove that diagonals of a parallelogram in the Euclidean plane split each other in halves.*

Consider the below parallelogram $ABCD$. Let $\overrightarrow{AB} = \overrightarrow{v}$ and $\overrightarrow{AC} = \overrightarrow{w}$. Let M be the midpoint of the diagonal AD. To prove the statement, we need to show that the diagonal CB also passes through M and that $|CM| = |MB|$.



According to the definition of vector addition (a.k.a. the parallelogram rule), $\overrightarrow{AD} = \overrightarrow{v} + \overrightarrow{w}$. Hence, $\overrightarrow{AM} = .5(\overrightarrow{v} + \overrightarrow{w})$.

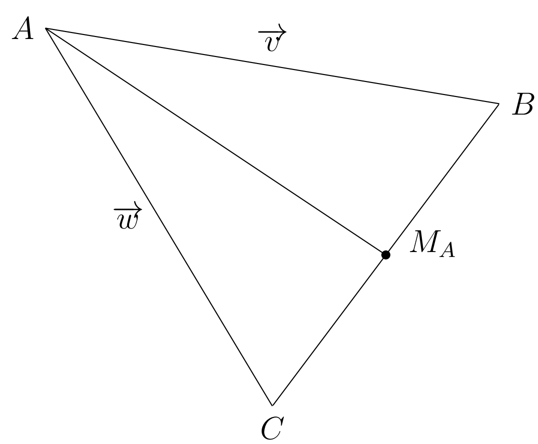
According to the definitions of an opposite vector and of vector addition, $\overrightarrow{CB} = \overrightarrow{v} - \overrightarrow{w}$. Hence, the vector that originates at C and terminates at the midpoint of the diagonal CB is $.5(\overrightarrow{v} - \overrightarrow{w})$. Let us add up $\overrightarrow{w}$ and this vector.

$$\overrightarrow{w} + .5(\overrightarrow{v} - \overrightarrow{w}) = .5(\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{AM}$$

In other words, if we first walk along the vector $\overrightarrow{AC}$ and then continue along the vector that originates at C and terminates at the midpoint of the diagonal CB, we end up at M, the midpoint of the diagonal AD. Therefore, the midpoints of the diagonals coincide. Q.E.D.

Following Archimedes of Syracuse, we have used geometry of weights to prove that all the three medians of a triangle intersect at one point that splits each median in the ratio $2:1$ counting from the vertex.[1] In the following sequence of problems, we will re-discover this wonderful fact using vector algebra.

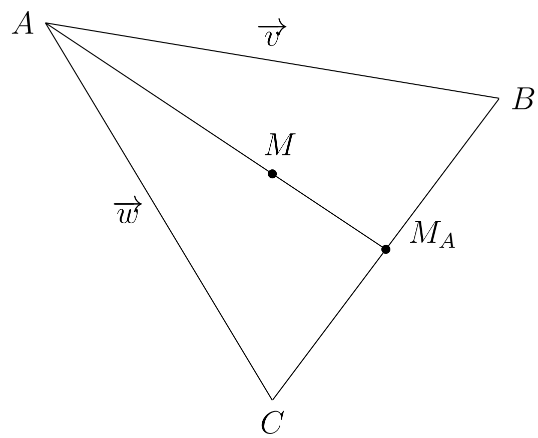
Problem 7 *Consider the triangle ABC below. Let $\overrightarrow{AB} = \overrightarrow{v}$ and $\overrightarrow{AC} = \overrightarrow{w}$. Let M_A be the midpoint of the side BC.*



Use the parallelogram rule to find the numbers a and b such that $\overrightarrow{AM_A} = a\overrightarrow{v} + b\overrightarrow{w}$. In other words, express $\overrightarrow{AM_A}$ as a linear combination of $\overrightarrow{v}$ and $\overrightarrow{w}$.

[1] http://www.math.ucla.edu/~radko/circles/lib/data/Handout-510-612.pdf

Problem 8 *Let M be a point of the median AM_A such that $|AM| = 2|MM_A|$.*

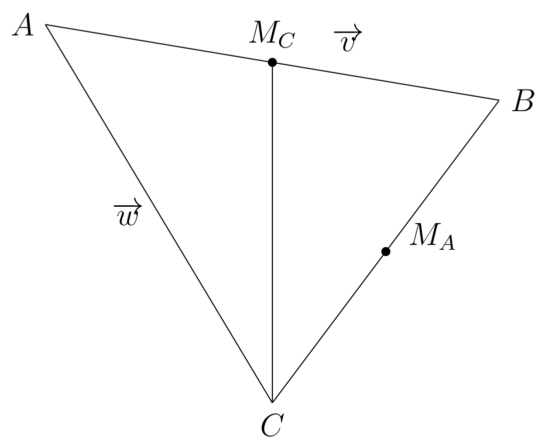


Express $\overrightarrow{AM}$ as a linear combination of $\overrightarrow{v}$ and $\overrightarrow{w}$. Simplify the coefficients of the expression, the numbers a and b such that $\overrightarrow{AM} = a\overrightarrow{v} + b\overrightarrow{w}$, as much as possible.

Let M_C (on the picture below) be the midpoint of the side AB. We need to show that

1. the line CM_C passes through M; and
2. $|CM| = 2|MM_C|$.

Problem 9 *Express* $\overrightarrow{CM_C}$ *as a linear combination of* $\overrightarrow{v}$ *and* $\overrightarrow{w}$.



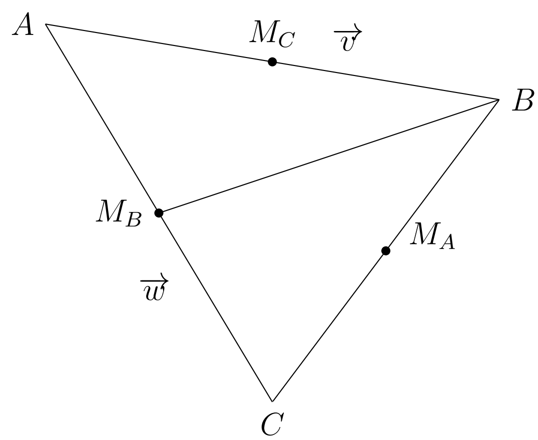
Problem 10 *Represent the vector*

$$\overrightarrow{w} + \frac{2}{3}\overrightarrow{CM_C}$$

as a linear combination of $\overrightarrow{v}$ *and* $\overrightarrow{w}$. *Compare the result to* $\overrightarrow{AM}$.

Let M_B be the midpoint of the side AC.

Problem 11 *Express $\overrightarrow{BM_B}$ as a linear combination of $\overrightarrow{v}$ and $\overrightarrow{w}$.*



Problem 12 *Represent the vector*

$$\overrightarrow{v} + \frac{2}{3}\overrightarrow{BM_B}$$

as a linear combination of $\overrightarrow{v}$ and $\overrightarrow{w}$. Compare the result to $\overrightarrow{AM}$.

We have proven the median theorem. However, a double-check never hurts.

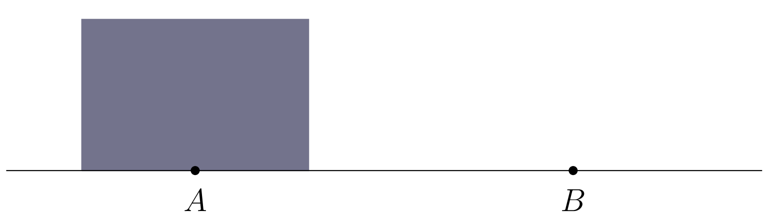
Problem 13 *Represent the vector*

$$\frac{1}{2}\overrightarrow{w} + \frac{1}{3}\overrightarrow{M_B B}$$

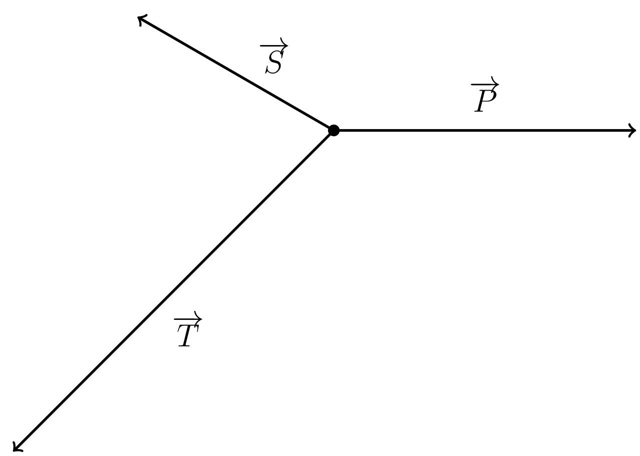
as a linear combination of $\overrightarrow{v}$ *and* $\overrightarrow{w}$*. Compare the result to* $\overrightarrow{AM}$*.*

As the following problem shows, vectors are a great tool not only in mathematics, but also in physics.

Problem 14 *You need to slide a heavy box over the floor from point* A *to point* B*. The box is about twice as short as you are. Which way is easier, to push or to pull? Why?*



Problem 15 *The dot on the picture below represents a spaceship. (Compared to the vastness of space, spaceships do look like dots.) There are three forces acting on the ship.* $\overrightarrow{T}$ *is the thrust of the ship's engine.* $\overrightarrow{P}$ *is the gravitational pull of the neighbouring planet.* $\overrightarrow{S}$ *is the gravitational pull of the planet's home star. You are the captain. Use a compass and a ruler to figure out where the resulting force would steer the ship.*



Next time, we will use vectors to formulate and study the laws of Newtonian mechanics, and more. In the meantime...

Problem 16 *Sum up all the integers from one to a thousand.*

Problem 17 *Do the same for the octals. Write down the answer in both the octal and the decimal form. Indicate the base by a subindex.*