Sequences

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Instructions: Each problem is labeled with 1-3 stars, 1 star being easiest and 3 stars being hardest. You should do (or at least be very confident that you know how to do) all of the 1 and 2 star problems in a section before moving on to the next one.

Section A

Evaluate the following limits, or write DNE if the limit does not exist.

- $(*)$ $\lim_{n\to\infty} \frac{5}{n}$
- $(*)$ $\lim_{n\to\infty} \frac{n^2}{3n^2-4}$
- $(*)$ $\lim_{n\to\infty} \frac{\cos(n)}{n}$
- $(*)$ $\lim_{n\to\infty} (-1)^n$
- $(*)$ $\lim_{n\to\infty} 9^{9^{9^9}} - n$
- $(**)$ $\lim_{n\to\infty} \frac{2^n}{n!}$
- $(**)$ $\lim_{n\to\infty} \frac{n^n}{n!}$
- $(**)$ $\lim_{n\to\infty} \sqrt{n^2+1} - n$
- $(**)$ $\lim_{n\to\infty} n^{(-1)^n - 1}$
- $(***)$ $\lim_{n\to\infty} \frac{n}{100} - \left\lfloor \frac{n}{100} \right\rfloor$
  ($\lfloor y \rfloor$ is defined as the greatest integer $\leq y$)
- $(***)$ $\lim_{n\to\infty} 10^{1/n}$
- $(***)$ $\lim_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

Section B

For each of the following limits, use the $\epsilon - N$ definition to either prove what the limit is or prove that it doesn't exist.

- $(*)$ $\lim_{n\to\infty} \frac{5}{n}$
- $(*)$ $\lim_{n\to\infty} \frac{1}{n^2}$
- $(*)$ $\lim_{n\to\infty} \frac{2n}{n+1}$
- $(*)$ $\lim_{n\to\infty} (-1)^n$
- $(**)$ $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$
- $(**)$ $\lim_{n\to\infty} \cos\left(\frac{n\pi}{10}\right)$
- $(**)$ $\lim_{n\to\infty} a_n$ where $a_n = \begin{cases} \frac{(-1)^{n/2}}{2^n} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$
- $(**)$ $\lim_{n\to\infty} 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^n}$
- $(***)$ $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{\sin n}{n+1}\right)$
- $(***)$ $\lim_{n\to\infty} \frac{(1/2)^n}{n^2}$

Section C

In this section, all proofs must either use the $\epsilon - N$ definition or cite a previous theorem that you have proven using the $\epsilon - N$ definition.

Exercise 1 ($*$)**.** Prove that if (a_n) and (b_n) are sequences with $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} (a_n + b_n) = a + b$.

Exercise 2 ($*$)**.** (a) Prove that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

(b) Is it true that if $\lim_{n\to\infty} |a_n| = 1$, then $\lim_{n\to\infty} a_n = 1$?

Exercise 3 ($*$)**.** (a) Suppose that $a_n \le s_n \le b_n$ for all n and that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$. Prove that $\lim_{n\to\infty} s_n = L$. (This is called the "squeeze theorem")

(b) Use the squeeze theorem to show $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

The following definitions may be useful for the rest of the problems.

Definition 1 (Infinite limits)**.** A sequence (s_n) is said to *diverge to* ∞, denoted $\lim_{n\to\infty} s_n = \infty$, if for all $M > 0$ there exists an N such that for all $n > N$, $s_n > M$.

Definition 2 (Upper and lower bounds)**.** Let A be any set of real numbers. A number B is an *upper bound* (resp. *lower bound*) for A if $a \le B$ (resp. $a \ge B$) for all $a \in A$. If such a B exists, A is said to be *bounded above* (resp. *bounded below*).

Definition 3 (Supremum and infimum)**.** Let A be a bounded set of real numbers. The *supremum* of A (or the *least upper bound* of A) is defined to be the unique number α with the following two properties:
(1) α is an upper bound for A.
(2) If β is an upper bound for A, then $\alpha \le \beta$.

The *infimum* of A (or the *greatest lower bound* of A) is defined to be the unique number α with the following two properties:
(1) α is a lower bound for A.
(2) If β is a lower bound for A, then $\alpha \ge \beta$.

The supremum and infimum are denoted $\sup A$ and $\inf A$ respectively.

Exercise 4 ($**$)**.** Let a_n and b_n be sequences such that $a_n \le b_n$. If $\lim_{n\to\infty} a_n = \infty$, prove that $\lim_{n\to\infty} b_n = \infty$.

Exercise 5 ($**$)**.** Let (s_n) be a monotonically increasing sequence, i.e. $s_n \le s_{n+1}$ for all n. Suppose also that (s_n) is bounded above. Prove that $\lim_{n\to\infty} s_n$ exists.

Exercise 6 ($**$)**.** For a sequence (s_n), define the "running averages" sequence

$$\sigma_n := \frac{1}{n}(s_1 + s_2 + \ldots + s_n).$$

(a) Prove that if $\lim_{n\to\infty} s_n = L$, then $\lim_{n\to\infty} \sigma_n = L$ as well.

(b) Show that (σ_n) may converge even if (s_n) doesn't converge.

Exercise 7 ($***$)**.** For a sequence (s_n), define

$$\limsup_{n\to\infty} s_n := \lim_{n\to\infty} \left(\sup_{m>n} s_m \right), \qquad \liminf_{n\to\infty} s_n := \lim_{n\to\infty} \left(\inf_{m>n} s_m \right).$$

(a) Prove that the limits above either exist or equal $\pm\infty$.

(b) Prove that (s_n) converges if and only if $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n$.