

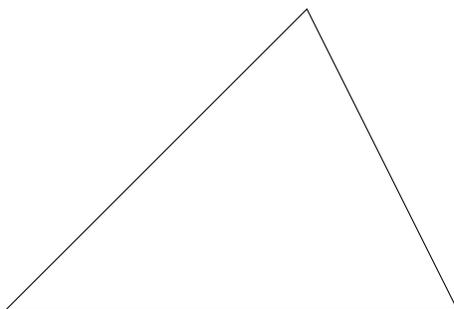
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Constructing Triangles.

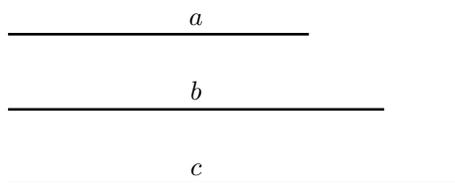
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The word *polygon* means *multi-angled* in the language of Euclid, the ancient Greek. The simplest possible polygon in the Euclidean plane is a triangle, a polygon with three vertices, sides, and angles.



Let us learn various ways to construct a triangle using a compass and a ruler.

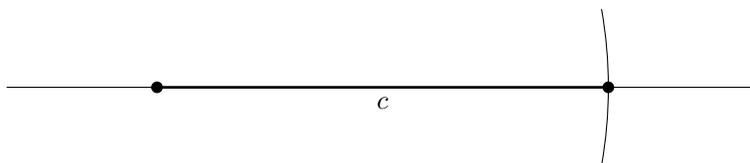
Example 1 *Using a compass and a ruler, draw a triangle with the given sides a , b , and c .*



Step 1. Draw a straight line and mark a point on it.

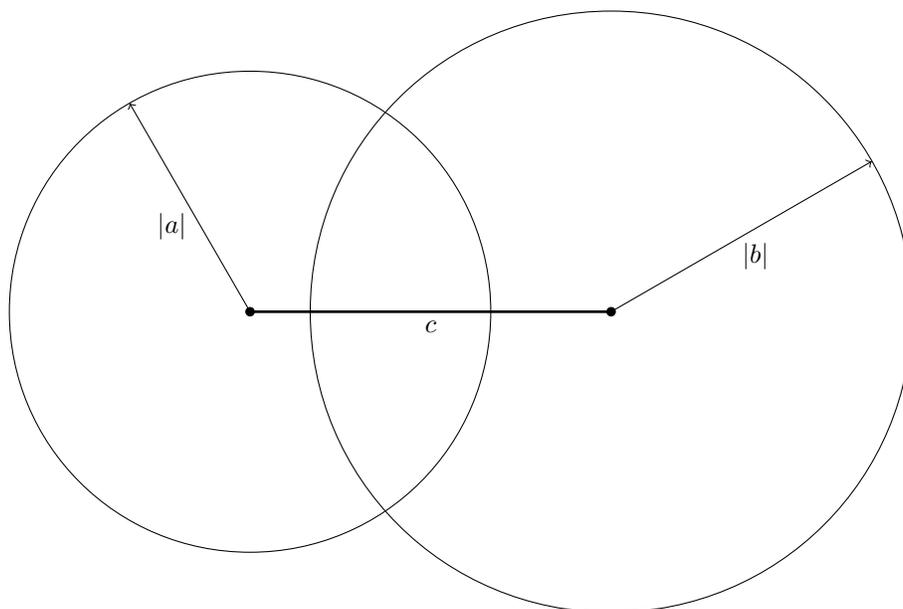


Step 2. Measure side c with a compass, place the needle at the marked point on the line, and mark side c on the line.

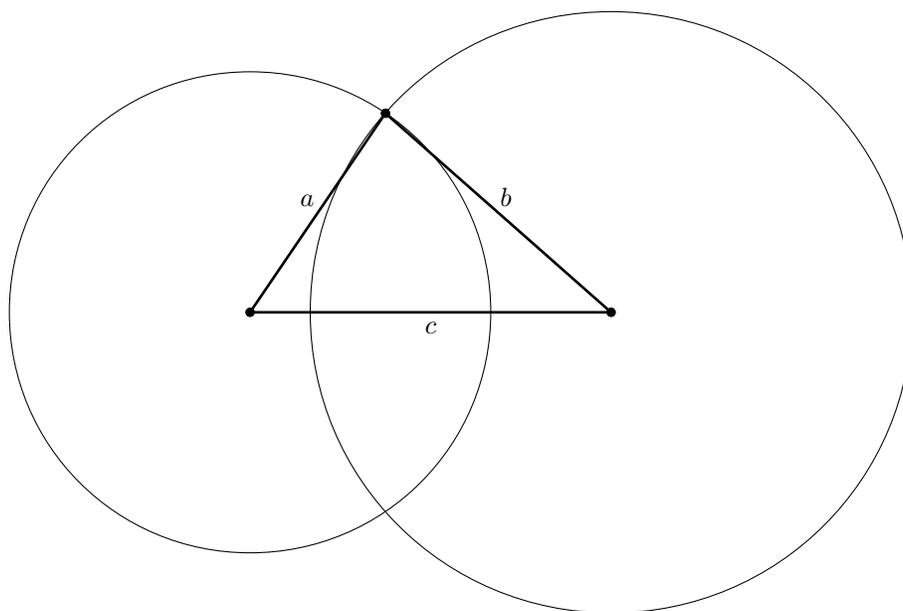


We will use straight parenthesis to denote the length of a straight line segment. For example, the length of the side a is $|a|$.

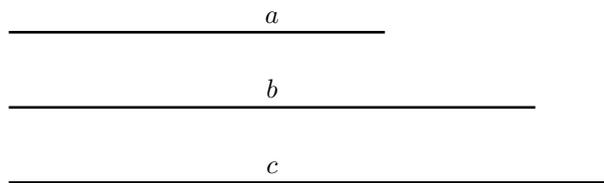
Step 3. Now it is time to recall the definition of a circumference. The points of the plane having the distance $|a|$ from the left end of side c form a circumference of radius $|a|$ centered at the left node of the above picture. Similarly, the points having the distance $|b|$ from the right end of c belong to the circumference of radius $|b|$ centered at the right node.



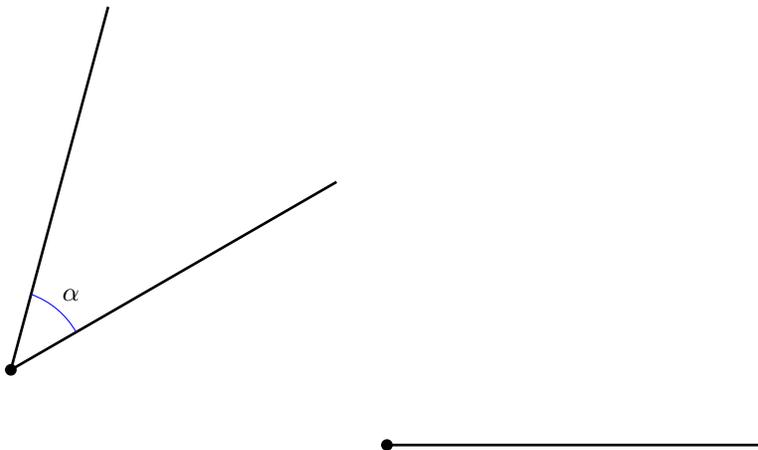
Step 4. The circumferences intersect at two points. Each of them has the distance $|a|$ from the left end of the side c and the distance $|b|$ from the right. We can pick either one as the third vertex of the triangle.



Problem 1 *Use a compass and a ruler to construct a triangle having the following sides.*



Example 2 *Using a compass and a ruler, draw a given angle α having a given ray as its side.*

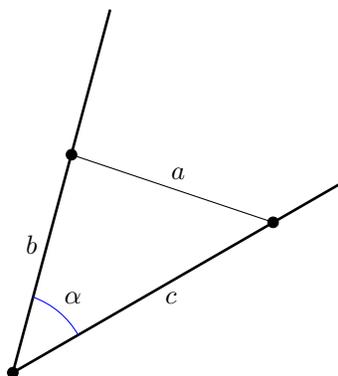


Below, you will see two different solutions to this problem, each having its own merit.

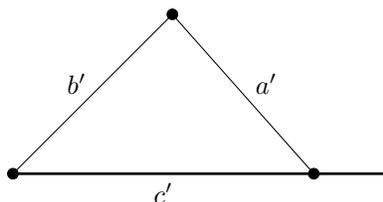
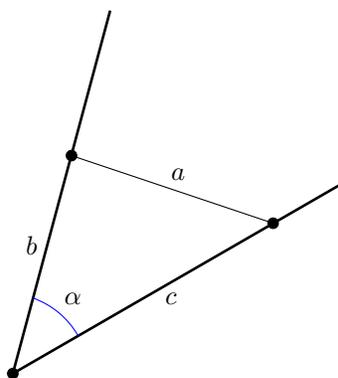
Solution 1. *The following approach is widely used in Mathematics. We need to solve a problem, but we don't know how. Instead, let us solve a problem that we know how to solve. In this particular case, we already know how to construct a triangle with given side lengths. Instead of solving the original problem, let's do that!*

The word “auxiliary” means “providing supplementary or additional help and support” as in an “auxiliary nanny”, a nanny occasionally employed in addition to the main one. As many more words used in science, this one originates from Latin, the language of the ancient Rome. Its progenitor, the Latin word “auxilium” means “help”.

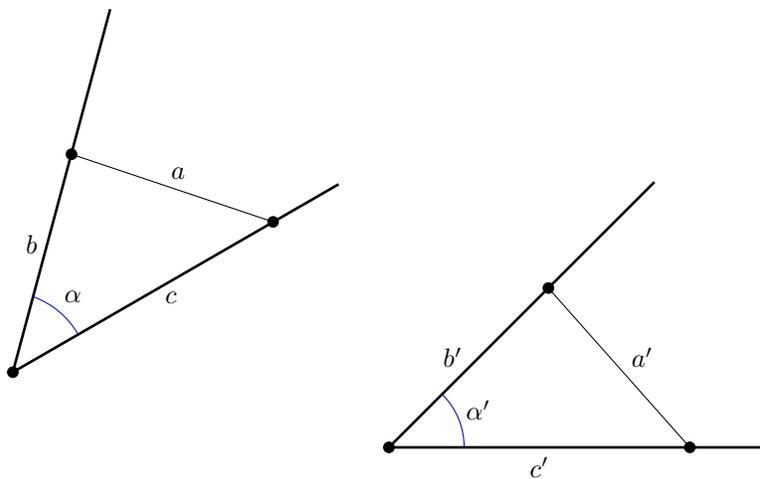
Step 1. Let us draw an auxiliary triangle having α as its angle. Traditionally, the side opposite to α is called a .



Step 2. Using the method of Example 1, let us construct the triangle having the sides a' , b' , and c' so that $|a'| = |a|$, $|b'| = |b|$, $|c'| = |c|$, c' goes along the given ray and the left end of c' coincides with the ray's vertex.

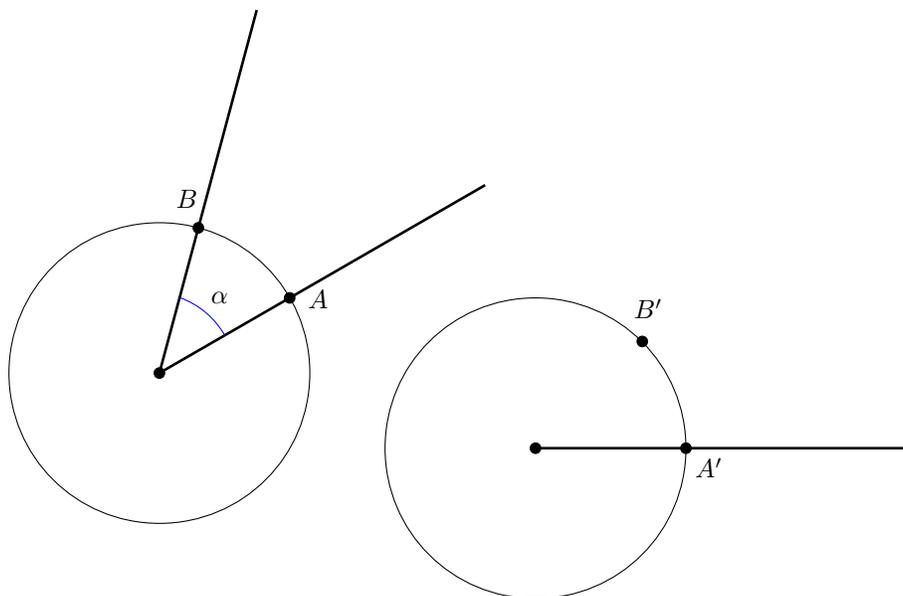


Since the constructed triangles are equal, the angle α' opposite to the side a' must be equal to the angle α opposite to the side a .

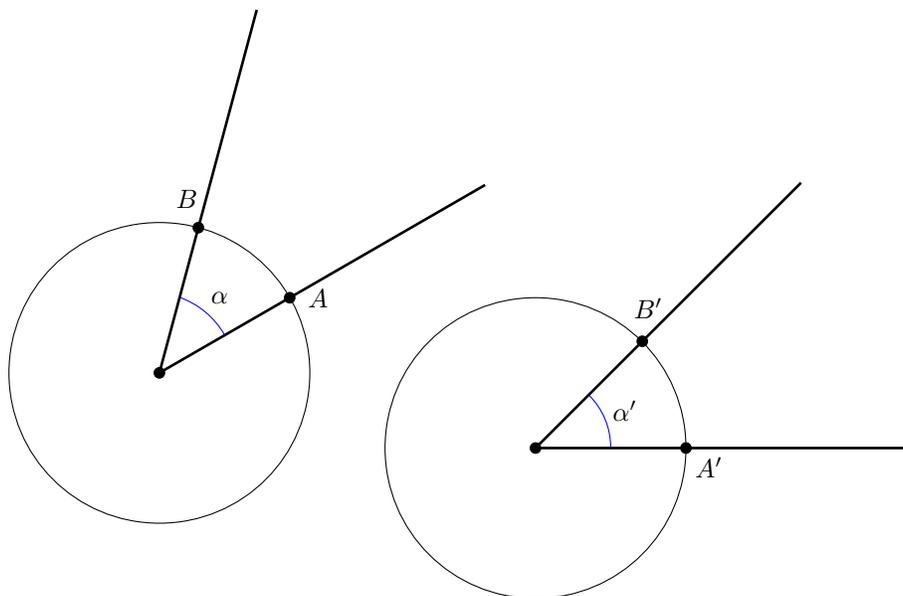


Solution 2. *Let us draw a circumference centered at the vertex of the original angle. The sides of the angle mark two points, A and B , on the line.*

Let us draw another circumference of the same radius centered at the vertex of the given ray. Let us call A' the point where the circumference intersects the ray. Let us measure the distance between the points A and B with a compass. Let us further stick the compass's needle at A' and mark the point B' lying on the second circumference such that $|AB| = |A'B'|$.

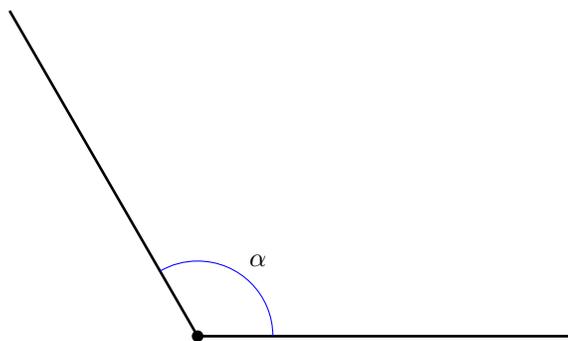


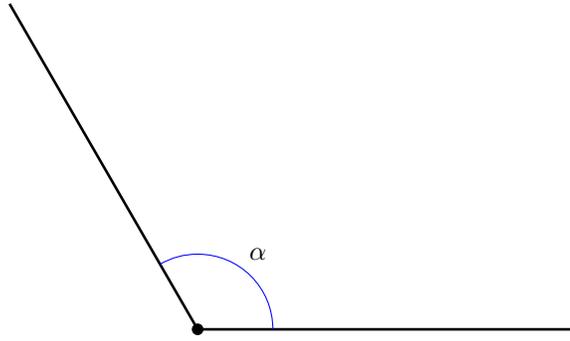
The last thing to be done is to draw the ray originating at the center of the second circumference and passing through B' .



To prove that $\alpha = \alpha'$, consider the translation of the plane (a move of the plane parallel to itself) that shifts the center of the second circumference to the center of the first. This move will make the circumferences coincide. Let us further rotate the second angle until A' coincides with A . Since $|AB| = |A'B'|$, this move will make B' coincide with B as well. Thus, the angles α and α' are equal.

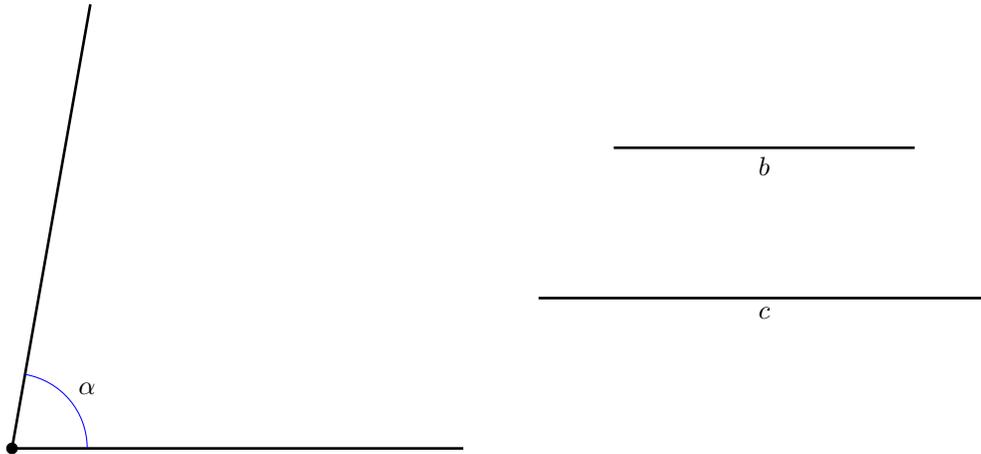
Problem 2 *Using a compass and a ruler, construct an angle equal to the angle α below in two different ways. Use an auxiliary triangle on this page and the circumferences on the next one.*





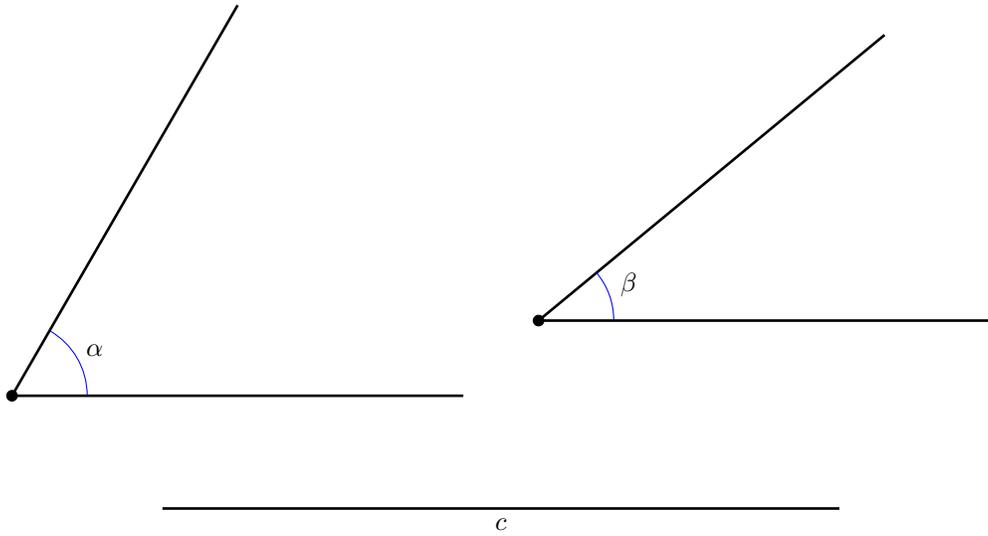
The word “adjacent” means “lying near to” as in “adjacent rooms” or “the houses adjacent to the park”. It was inherited from Latin without a change in spelling.

Problem 3 *On the next page, draw a triangle with the angle α and adjacent sides b and c given below. In this case, the word “adjacent” means that the vertex of α is an endpoint of the sides b and c .*



Hint: begin with drawing the angle. If you choose the auxiliary triangle method for constructing the latter, you can use not some arbitrary sides b and c , but the given ones right away!

Problem 4 Draw a triangle with the side c and adjacent angles α and β given below. In this case, the word “adjacent” means that the endpoints of c are the vertices of α and β .



Note that by carrying out the above constructions, we just have proven the following very important theorem.

Theorem 1 *Two triangles in the Euclidean plane are equal if either of the following holds.*

- *Their side lengths are pairwise equal.*

$$|a| = |a'|, \quad |b| = |b'|, \quad |c| = |c'|$$

- *They have an angle of equal size, and the lengths of the sides adjacent to the equal angles are pairwise equal.*

$$\alpha = \alpha', \quad |b| = |b'|, \quad |c| = |c'|$$

- *They have a side of equal length, and the adjacent angles are pairwise equal.*

$$|c| = |c'|, \quad \alpha = \alpha', \quad \beta = \beta'$$

Problem 5 *On the next page, construct a triangle with the angle α given below as well as the side b adjacent to α and the side a opposite to the angle.*

