CRACKING THE 15 PUZZLE - PART 4: TYING EVERYTHING TOGETHER

BEGINNERS 02/21/2016

Review

Recall from last time that we proved the following theorem:

Theorem 1. *The sign of any transposition is* -1.

Using this theorem, we were able to solve the last few questions in the handout. We will go over these as a class:

Problem 1. Suppose we have a 15 puzzle where the distance between the blank square and the lower-right corner is an odd number. Is the number of transpositions we need to apply to the puzzle to move the blank square to the lower-right corner odd or even?

Problem 2. Suppose we have a 15 puzzle with an odd number of inversions. How many transpositions do we need to apply to the puzzle so that there are no inversions?

Problem 3. Do you think a solution for Loyd's puzzle, as shown below exists? Explain your answer.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Based off of our answer in Problem 3, we can now construct and prove a second theorem:

Theorem 2. *Configurations of the 15 puzzle have "similar parity" when the number of inversions and the taxicab distance from the blank space to the lower right corner are both odd or both even. Configurations of the 15 puzzle have "opposite parity" otherwise. Configurations of the 15 puzzle with opposite parities cannot be solved.*

Proof. Configurations of the 15 puzzle with opposite parity cannot be solved.

□

Are We Done? (Some Questions About Logic)

Problem 4. We know from Theorem 2 that 15 puzzles with configurations of opposite parity cannot be solved. Do you think we can now determine whether or not there's a solution for all configurations of the 15 puzzle? Why or why not?

Regardless of your answer, we need some logic to figure out what the right answer to Problem 4 is.

We use the symbol $\implies$ to mean "implies that".

If we had two statements, P and Q, $P \implies Q$ means that IF P is true, THEN Q is true.

Example. Theorem 2 can be thought of as an if-then statement, where P and Q are the following:

$$P: \text{ "The configuration of the 15 puzzle has opposite parity."}$$

$$Q: \text{ "The configuration of the 15 puzzle cannot be solved."}$$

This gives the following statement:

"The configuration of the 15 puzzle has opposite parity $\implies$ the configuration of the 15 puzzle cannot be solved."

Which is equivalent to

"If the configuration of the 15 puzzle has opposite parity, then the configuartion of the 15 puzzle cannot be solved."

Example. Some more if-then statements are shown below:

- It is 100 degrees outside $\implies$ It is hot outside
- Goldie is a dog $\implies$ Goldie is an animal
- It is raining $\implies$ There is traffic

Problem 5. Come up with your own if-then statement below.

The *contrapositive* of an if-then statement is when we negate each statement and turn the if-then statement around. We use the ! symbol to mean "not".

So the contrapositive of $P \implies Q$ is $!Q \implies !P$.

Problem 6. The contrapositive to the statements given in the above examples are shown below. If the statements in the examples above are true, are each of the contrapositives shown below true?

- It is not hot outside $\implies$ It is not 100 degrees outside
- Goldie is not an animal $\implies$ Goldie is not a dog
- There is no traffic $\implies$ It is not raining

Logically, all statements are equivalent to their contrapositive.

Problem 7. What is the contrapositive to Theorem 2? (Check your answer with your assistant instructor to make sure it's correct.)

The *converse* of an if-then statement is when we turn the if-then statement around without negating each statement.

So the converse of $P \implies Q$ is $Q \implies P$.

The converse of the statement is NOT always true.

Problem 8. The converse to the statements given in the above examples are shown below. If the statements in the examples above are true, are each of the converses shown below true? Give a counterexample to show how the statements below could be wrong.

- It is hot outside $\implies$ It is 100 degrees outside
- Goldie is an animal $\implies$ Goldie is a dog
- There is traffic $\implies$ It is raining

Problem 9. What is the converse to Theorem 2? (Check your answer with your assistant instructor to make sure it's correct.)

Because not all converses are true, we have to prove that the converse is true in order to use it.

Problem 10. If we knew that the configuration of the 15 puzzle had similar parity, could we conclude that the 15 puzzle was solvable? Explain your answer.

Theorem 3. *Any configuration of the 15 puzzle with similar parity is solvable.*

Because Theorem 3 is the converse of Theorem 2, we have to solve it in order to use it. Theorem 3 is not hard to prove using mathematical induction. However, we are not going to do it at the moment as induction proofs are a little bit beyond what we have learned. For now, we can take it as a fact without proving it.