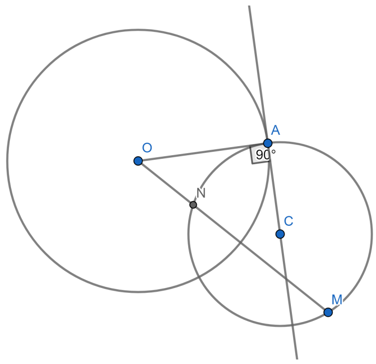
Hyperbolic Geometry Solutions

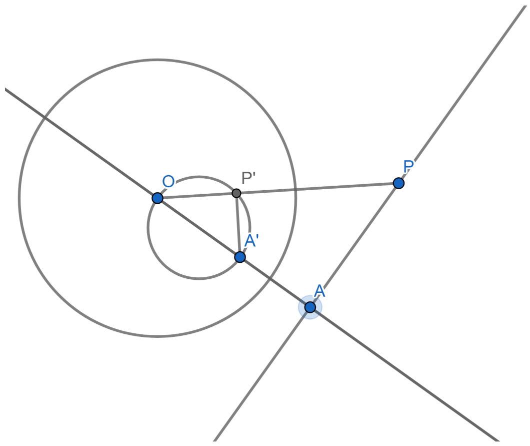
LAMC

November 11, 2018

Problem 1.3. b) By Power of a Point (Problem 1.1) we have $|ON| \cdot |OM| = |OA|^2 = r^2$, so each orthogonal circle is fixed under inversion.



c) Lines through O, the center of ω, are mapped to themselves. Now consider the case when the line ℓ is outside the circle of inversion. Drop the foot from O to A. Invert A

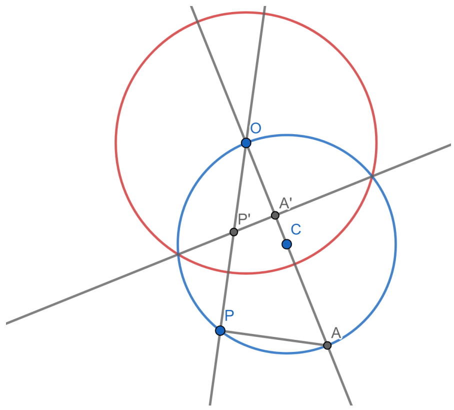


about the circle to obtain A'. Then construct the circle Γ going through O and A' with radius $|OA'|/2$. Let P be a point on ℓ and let P' be the intersection of the line through OP with Γ. We must show $|OP'||OP| = r^2$. We have $OP'A' \sim OAP$ so

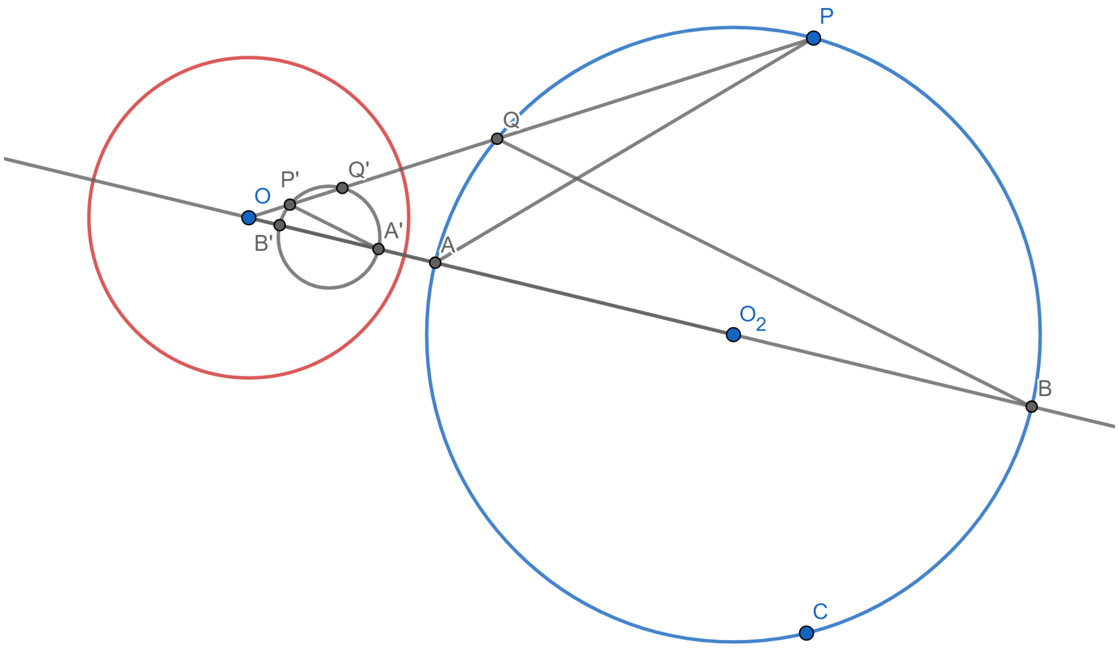
$$\frac{|OP'|}{|OA|} = \frac{|OA|}{|OP|} \implies |OP'||OP| = |OA|^2$$

The same argument shows a circle through the center of ω is mapped to a line outside ω.

Now suppose the line intersects ω. The same argument works.



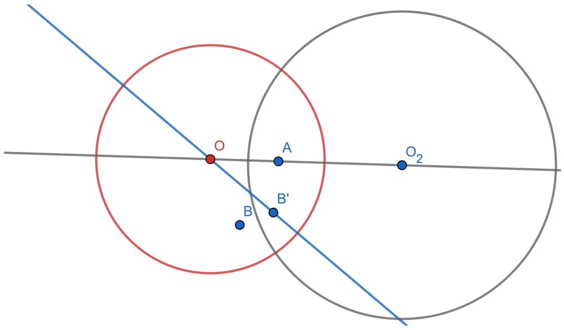
Now consider the case of a circle Γ outside of the circle of inversion ω.



Draw the line through the centers O and O_2. Let A and B be the intersections of this line with Γ. We construct a circle Γ' as follows: Invert A and B about ω to A' and B' respectively and construct the circle through B' and A' with radius $|B'A'|/2$. Now let P be a point on Γ. It suffices to show the inversion P' of P is on Γ'. To that end construct the line through O and P. Let Q be its intersection with Γ and Q' its intersection with Γ'. Now $OA'P' \sim OBQ$. Note that $\angle APQ = \angle OBQ$ since they subtend the same arc. Thus $OA'P \sim OPA$, so we get

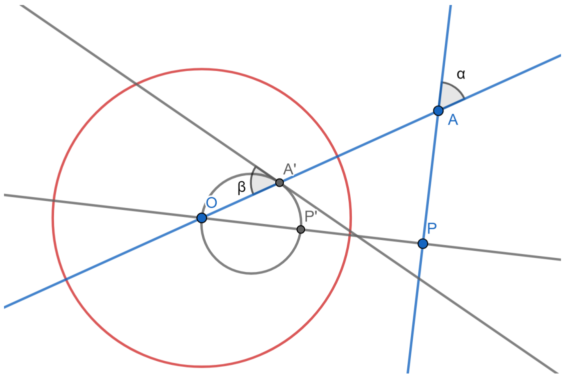
$$\frac{|OA'|}{|OP'|} = \frac{|OP|}{|OA|} \implies |OP||OP'| = |OA||OA'| = r^2$$

Problem 1.5. By 1.4 we can construct a circle Γ so that the inversion of A about Γ is O. Now invert B about Γ and let its image be B'. Construct the line through O and B'. Inverting this line about Γ gives us a circle going through A and B.



This circle is orthogonal to ω because the line OB' is orthogonal to ω.

Problem 1.6. First suppose the first line ℓ_1 goes through O. Let the second line ℓ_2 intersect ℓ_1 outside the circle at an angle α. The inversion of ℓ_2 about ω is a circle Γ passing through O. The inverted angle β is the angle of intersection of OA and the tangent to Γ at A'. We have $OA'P' \sim OPA$ so $\angle OAP = \angle OP'A' = \alpha$. But β subtends the same arc of Γ' so $\alpha = \beta$ (this theorem works even for tangent lines).



For case where ℓ_1 and ℓ_2 do not go through the origin, just note that when we invert them about ω and construct the tangent lines, these tangent lines are parallel to ℓ_1 and ℓ_2 respectively.

Problem 1.8. Assume A and C are not collinear with O (this case follows easily). If A' and C' are the images of A and C respectively, then $OAB \sim OB'A'$. Therefore $\frac{|AC|}{|A'C'|} = \frac{|OA|}{|OC'|}$ The same argument gives us the analgous statement for BC, BD, and AD. Thus the cross ratio becomes

$$\frac{|AC|}{|A'C'|}\frac{|B'C'|}{|BC|}\frac{|BD|}{|B'D'|}\frac{|A'D'|}{|AD|} = 1$$

Rearranging we obtain $[A, B; C, D] = [A', B'; C', D']$.

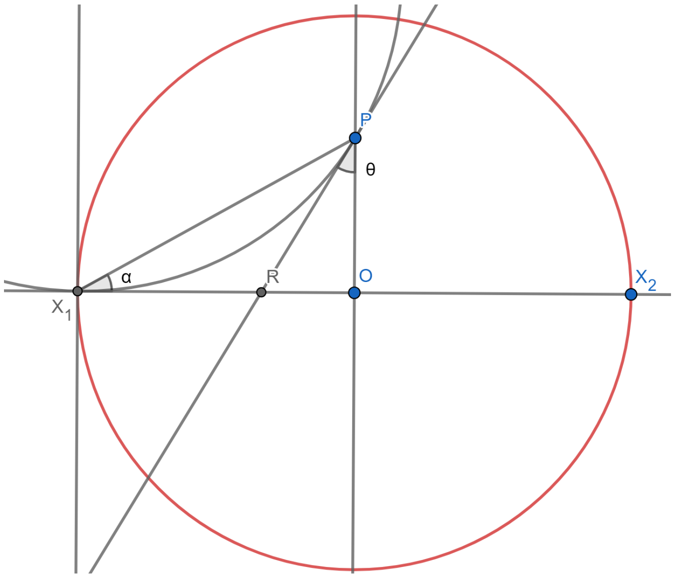
Problem 2.3. e) We assume P and Q are ordered so that $|OP| \leq |OQ|$ and $|AP| \geq |AQ|$. We have

$$\log\left(\frac{|AP||OQ|}{|OP||AQ|}\right) = \log\frac{1+r}{1-r}$$

Remark: the last quantity is actually given in terms of the hyperbolic tangent function $2\tanh^{-1} r$ defined by

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1-e^{-2x}}{1+e^{-2x}}$$

Problem 2.13. WLOG we take ℓ to be the diameter of $\mathbb{D}$. X_1RP is isosceles since



its sides are the tangents to the circle going through P. Since $\angle X_1OP = \pi/2$ we have $2\alpha + \theta = \pi/2$ or $\alpha = \pi/4 - \theta/2$. Now we calculate the Euclidean length $|OP|$

$$|OP| = \tan\alpha = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}$$

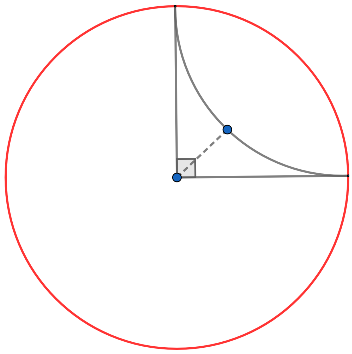
exponentiating and using 2.3 e) we get

$$e^d = \frac{1+\tan\alpha}{1-\tan\alpha} = \frac{1}{\tan(\theta/2)}$$

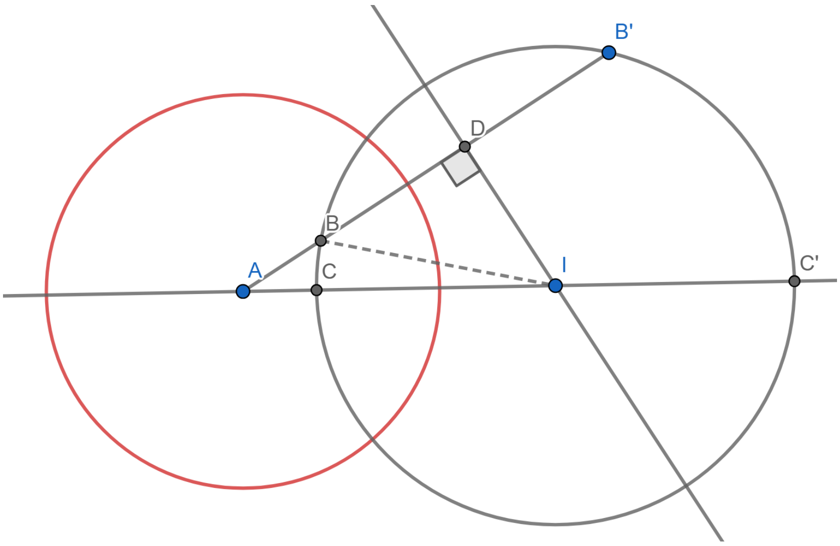
Hence $e^{-d} = \tan(\theta/2)$.

Problem 2.14. The upper bound is attained when two sides are radii. We can calculate the length of the perpendicular using Lobachevskii's formula with $\theta = \pi/4$

$$d = -\log\left(\tan\frac{\pi}{8}\right) = -\log\left(\tan(1-\sqrt{2})\right) = \log(1+\sqrt{2})$$



Problem 2.15. (c) Move A to the origin so that AB and AB are Euclidean line segments and CB is the arc of an orthogonal circle with center I. Problem 1.3 (a) shows that $|AB||AB'| = 1$ and $|AC||AC'| = 1$. By 2.5, the Euclean lenght of AB is $\tanh\frac{c}{2}$. Hence



$$|BB'| = |AB| - |AB'| = \tanh\frac{c}{2} - \frac{1}{\tanh\frac{c}{2}} = \frac{\operatorname{sech}^2\frac{c}{2}}{\tanh\frac{c}{2}} = \frac{2}{\sinh c}$$

where we have used the double angle formula $2\sinh x\cosh x = \sinh(2x)$. By a theorem from elementary geometry $\angle B = \angle BIB'/2 = \angle BID$. Now we have

$$\sin\angle B = \frac{|BD|}{|BI|} = \frac{|BB'|/2}{|CC'|/2} = \frac{\sinh b}{\sinh c}$$

Similarly we calculate

$$\cos\angle A = \frac{|AD|}{|AI|} = \frac{|AB| + |BB'|/2}{|AC| + |CC'|/2} = \frac{\tanh(c/2) + \frac{1}{\sinh c}}{\tanh(b/2) + \frac{1}{\sinh b}} = \frac{\tanh(c)}{\tanh(b)}.$$

Where we used the half angle formula

$$\tanh(x/2) = \frac{\cosh x - 1}{\sinh x}.$$

(d) We have

$$\begin{aligned}
1 = \sin^2\angle A + \cos^2\angle A &= \frac{\sinh^2(a)}{\sinh^2(c)} + \frac{\tanh^2(c)}{\tanh^2(b)} \\
\sinh^2 c &= \sinh^2 a + \cosh^2 c \cdot \tanh^2 b \\
\cosh^2 c &= \cosh^2 a + \cosh^2 c \cdot \tanh^2 b \\
\cosh^2 c \cdot \cosh^2 b &= \cosh^2 c \cdot \sinh^2 b + \cosh^2 a \cdot \cosh^2 b \\
\cosh^2 c &= \cosh^2 a \cdot \cosh^2 b
\end{aligned}$$

Which proves the theorem.

Problem 2.16. Let $a = re^{i\theta}$. Referring to Figure 2 from Problem 1.4, φ_a fixes the circle of inversion with center I and radius $\tan\theta$, where I is the inversion about $\mathbb{D}$ of a. the coordinates of I in complex notation is then $r^{-1}e^{i\theta}$. It is then routine to check that the circle

$$C = \left\{ z \in \mathbb{C} : \left| z - \frac{1}{r}e^{i\theta} \right| = \tan\theta \right\}$$

is preserved under φ_a.

$$\begin{aligned}
\varphi_a(r^{-1}e^{i\theta} + e^{i\alpha}\tan\theta) &= \frac{e^{i\theta}(r^{-1} + r) + e^{i\alpha}\tan\theta}{-re^{-i\theta}e^{i\alpha}\tan\theta} \\
&= -e^{-i\alpha}\left(e^{2i\theta}\frac{1-r^2}{r}\frac{1}{\tan\theta} \right) + \frac{1}{r}e^{i\theta} \\
&= -e^{-i\alpha}\left(\tan\theta e^{i2\theta} \right) + \frac{1}{r}e^{i\theta} \qquad \left(\text{since } \tan\theta = \frac{\sqrt{1-r^2}}{r} \right)
\end{aligned}$$

which is again in C. However, φ_a is not exactly a circle inversion. For example, the inversion about a circle at the origin with radius R is given by the conjugate of complex inversion: $f(z) = R^2/\bar{z}$.