

CRACKING THE 15 PUZZLE - PART 1: PERMUTATIONS

BEGINNERS 01/24/2016

The ultimate goal of this topic is to learn how to determine whether or not a solution exists for the *15 puzzle*.

The puzzle consists of a 4×4 frame randomly filled with 15 squares numbered one through fifteen. The objective is to slide the squares so that they are in the proper order, as shown in the picture below:



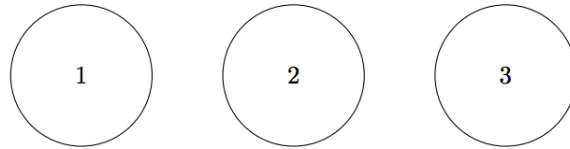
The mathematical foundation of the solution relies on the theory of *permutations* and *taxicab distance*. Permutations not only help unravel the puzzle, but can also be handy in a wide variety of applications from card tricks to probability. Taxicab distance is a way of measuring distance on the plane by going only in the vertical and horizontal direction.

Part 1 of this topic will focus on what permutations are, how they are represented, the multiplication of permutations and the inverse of permutations.

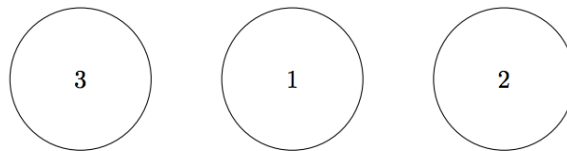
Permutations and Their Representations

Consider a set of marbles numbered 1 through n . Originally, the marbles are lined up in the order given by their numbers.

The following picture shows an example with $n = 3$:



Then the marbles are reshuffled in a different order:



We say that the reshuffled set of marbles is a *permutation* of the first set of marbles. A *permutation* is an operation of the elements of any given set that shuffles the order of the elements.

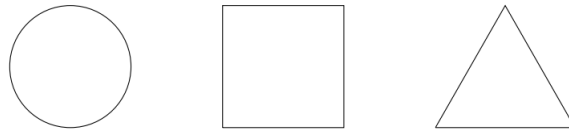
The permutation shown in the example above is represented by the notation below:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

One way to understand the notation above is the following:

- The top row assigns a number to each element in a set
- The bottom row shows how the numbered elements are positioned after we have applied the permutation to the set.

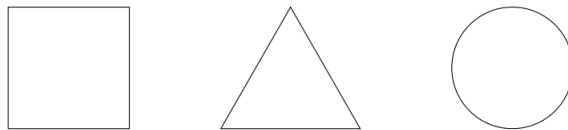
Instead of the numbered marbles, we can reshuffle distinguishable elements of any set. For example, let's consider the following geometric figures rather than the numbered marbles:



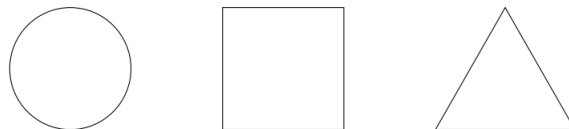
Then the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

will reshuffle the figures into the following order:



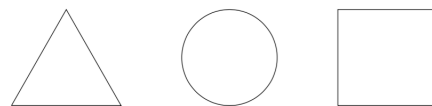
Problem 1. Suppose you're given the following set of geometric figures:



(1) Write down the permutations that correspond to the following pictures:



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

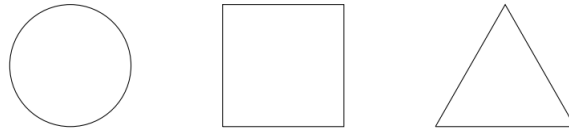


$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

(2) Draw the figures that correspond to the following permutations on the following set:



(a) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Note that the top row is not 1 2 3.

(c) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

Notice that the last permutation does not reshuffle anything at all. Permutations of this kind are typically denoted as e and are called *trivial*. A trivial permutation is still a permutation, and an important one!

Problem 2. Write down the trivial permutation for $n = 5$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

Multiplying Permutations

It is possible for us to combine, or *multiply*, permutations. The following problem will walk you through how this is done.

Problem 3. Suppose we want to first apply the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and then apply the permutation $\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ on the following set:



(1) First apply the $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ to the set above. Draw the resulting order below:

(2) Now apply $\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ to the ordered set you obtained in (1). Draw the resulting order below:

(3) We say that we have applied the product of δ and σ to the original set. Represent the product of δ and σ as a single permutation by comparing the ordered set you drew after applying both permutations to the original set:

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Note that if we're given a product of permutations such as $\delta \circ \sigma$, the permutation on the right (in this case, σ) is applied to the set first!

Problem 4. Find the permutation of $\sigma \circ \delta$. If needed, use the steps described in the examples above to help you.

$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

Is $\sigma \circ \delta = \delta \circ \sigma$?

We say that the multiplication of two permutations *commutes* if $\sigma \circ \delta = \delta \circ \sigma$. While some particular permutations may commute, multiplication of permutations in general is not a commutative operation.

Problem 5. Find the product $\delta \circ \sigma$ of the following two permutations:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

Remember to start with the permutation on the right.

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 6. Find the product $\sigma \circ \delta$ of the above permutations.

$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Do the permutations δ and σ commute?

Opposite/Inverse Permutations

Let δ and σ represent permutations. δ is *opposite* to σ if $\delta \circ \sigma = e$. In other words, δ undoes what σ does. Such a permutation is denoted as σ^{-1} and is called the *permutation opposite to sigma* or *sigma inverse*.

Problem 7. Find σ^{-1} for $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

(1) Suppose we are given the following set:



What will the set look like after applying σ to it? Draw the resulting ordered set below.

(2) Taking the second ordered set we drew, we now want to find the permutation σ^{-1} so that it will “undo” what σ did. In other words, we want to find the permutation that will transform the second ordered set back to the original ordered set.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Problem 8. Find σ^{-1} for $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Problem 9. Find σ^{-1} for $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

The above two problems are examples of two different non-trivial permutations that are self inverse. In other words, $\sigma^{-1} = \sigma$.

When dealing with numbers, the only self inverse numbers are -1 and 1 . Unlike numbers, there exist lots of different non-trivial self-inverse permutations.

Problem 10. Given any σ and σ^{-1} ,

(1) what is $\sigma \circ \sigma^{-1}$?

(2) What is $\sigma^{-1} \circ \sigma$?

(3) Do σ and σ^{-1} commute?

More Notation

You may have noticed that the first line of the notation we have used for writing down permutations is not necessarily needed. The permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

means that we shuffle the second element to the first position, the first element to the second position, etc. Without any loss of clarity, we can represent this permutation as

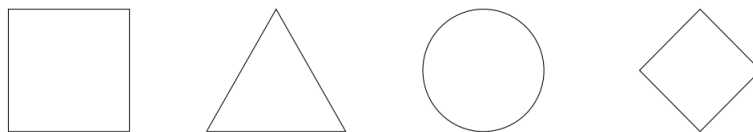
$$\sigma = (2 \ 1 \ 4 \ 3)$$

Problem 11. Rewrite the following permutations in our new notation:

$$(1) \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

$$(2) \sigma = \begin{pmatrix} 4 & 2 & 3 & 1 \\ 4 & 2 & 1 & 3 \end{pmatrix} \text{ Hint: Notice that the top row is not } 1 \ 2 \ 3 \ 4. \text{ Even though the top row is not necessarily constant, we can replace the numbers so that the top row becomes } 1 \ 2 \ 3 \ 4.$$

Problem 12. Apply the following permutations to the sequence of figures shown below:



$$(1) \sigma = (1 \ 3 \ 2 \ 4)$$

$$(2) \sigma = (4 \ 3 \ 1 \ 2)$$

Problem 13. What is the inverse of the following permutations? You can use the pictures you drew from Problem 13 to help you.

(1) $\sigma = (1 \ 3 \ 2 \ 4)$

(2) $\sigma = (4 \ 3 \ 1 \ 2)$

