PARITY, SYMMETRY, AND FUN PROBLEMS[1]

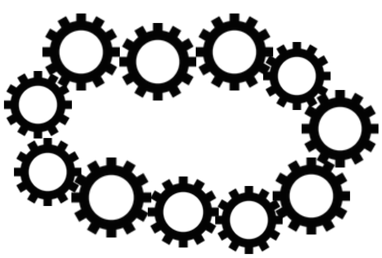
November 2, 2014

Warm Up Problems

Below are 11 numbers - six zeros and five ones. Perform the following operation: cross out any two numbers. If they were equal, write another zero on the paper. If they were not equal, write a one. Continue doing this until you are left with one number. What must the remaining number be? Why?

0 0 0 0 0 0 1 1 1 1 1

Eleven gears are placed on a plane, arranged in a chain as shown below. Can all gears rotate simultaneously? Explain your answer.



[1]**Problems taken from *Mathematical Circles (Russian Experience)* by Dmitri Fomin, Sergey Genkin, and Ilia Itenberg**

Chessboard and Checkerboard Problems

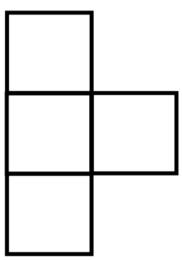
1. On a chessboard, a knight starts from square $a1$, and returns there after making several moves. Show that the knight makes an even number of moves. To help explain your answer, draw a picture.

2. Can a knight start at square $a1$ of a chessboard, and go to square $h8$, visiting each of the remaining squares exactly once on the way? Explain your answer.

3. Can a 5×5 checkerboard be covered by 1×2 dominoes? Explain.

4. A standard 8×8 chessboard has two diagonally opposite corners removed, leaving 62 squares. Is it possible for this chessboard to be covered by 1×2 dominoes? Explain.

5. Can one cover a 10×10 checkerboard using the piece shown below? Explain your answer.



6. Consider a 8×8 checkerboard with one of the corner tiles removed (so that there are a total of 63 tiles). Can one cover this checkerboard using dominoes of size 1×3? Explain.

Dominoes Problems

7. All of the dominoes are laid out in a chain so that the number of spots on the ends of the adjacent dominoes match.

(a) Complete the table below by listing out all of the dominoes.

00	11	22				
01	12					
02						
03						

(b) Write down how many times each of the numbers appear in the domino set.

Number	0	1	2	3	4	5	6
Frequency							

(c) If one end of the domino chain is a 5, what is at the other end? Explain.

8. In a set of dominoes, all those in which one square has no spots are discarded. Can the remaining dominoes be arranged in a chain? Explain.

Polygons and Axes of Symmetry

9. Given a convex 101-gon which has an axis of symmetry, prove that the axis of symmetry passes through one of its vertices.

   What can you say about a 10-gon with the same properties?

10. Twenty five checkers are placed on a 25×25 checkerboard in such a way that their positions are symmetric with respect to one of its diagonals. Prove that at least one of the checkers is positioned on that diagonal.

11. Let us now assume that the positions of the checkers in Problem 13 are symmetric with respect to both diagonals of the checkerboard. Prove that one of the checkers is placed in the center square.

Miscellaneous Problems

12. Katya and her friends stand in a circle. It turns out that both neighbors of each child are of the same gender. If there are five boys in the circle, how many girls are there?

13. Can one make change of a 25 ruble bill, using in all ten bills each having a value of 1, 3, or 5 rubles?

14. Pete bought a notebook containing 96 pages, and numbered them from 1 to 192. Victor tore out 25 pages of Pete's notebook, and added the 50 numbers he found on the pages. Could Victor have gotten 1900 as the sum?

15. The product of 22 integers is equal to 1. Show that their sum cannot be zero.

16. Can one form a "magic square" out of the first 36 prime numbers?

    A "magic square" is 6×6 array of boxes, with a number in each box, and such that the sum of the numbers along any row, column, or diagonal is constant.

17. The numbers 1 through 10 are written in a row. Can the signs "+" and "-" be placed between them, so that the value of the resulting expression is 0?

18. A grasshopper jumps along a line. His first jump takes him 1 cm, his second 2 cm, and so on. Each jump can take him to the right or to the left. Show that after 1985 jumps the grasshopper cannot return to the point at which he started.

19. Is it possible to arrange the numbers from 1 through 9 in a sequence so that there are oddly many numbers between 1 and 2, between 2 and 3, ... , and between 8 and 9? Explain your answer.