§2 The Poincaré Disc

The hyperbolic plane $\mathbb{H}^2$ can be modeled as the *Poincaré disc*, the unit ball in the complex plane

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

Points on the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ are called *ideal points*. We define *hyperbolic lines* to be circles orthogonal to $\mathbb{S}^1$ or diameters of $\mathbb{S}^1$. Lines are *parallel* if they do not intersect.

Definition 2.1. Let A, B be points in the Poincaré disc, and let P and Q be the ideal endpoints of the hyperbolic line joining A and B. The *hyperbolic distance* between A and B is defined as

$$d(A,B) = |\log([A,B;P,Q])| = \left|\log\left(\frac{|AP|\cdot|BQ|}{|AQ|\cdot|BP|}\right)\right|$$

Question 2.2. What are the isometries of the Poincaré disc (symmetries that don't change the hyperbolic distance)?

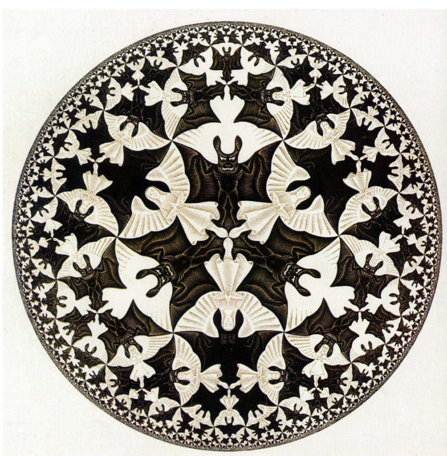

Figure 3: "Angel-devil" by M.C. Escher. Each devil has the same hyperbolic size!

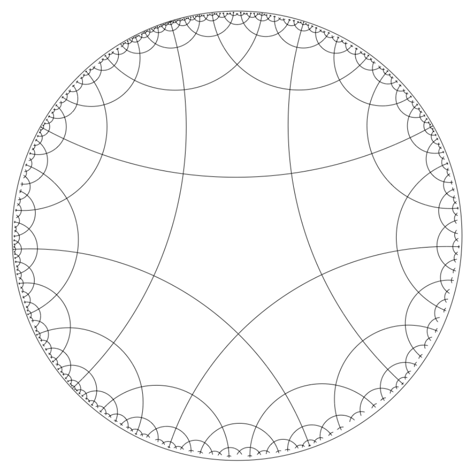

Figure 4: Tiling of hyperbolic plane by pentagons

Problem 2.3. (a) Show that $d(A,B) \geq 0$. When is the distance 0?

(b) Show that $d(A,B) = d(B,A)$

(c) If A, B, and C lie on the same hyperbolic line, show that $d(A,C) = d(A,B) + d(B,C)$. Show that the equality becomes $\leq$ if C is not on the same hyperbolic line.

Problem 2.4. Show that Axiom $5'$ is satisfied with our new definition of "line."

Problem 2.5. (a) What is the hyperbolic distance between the origin and $\frac{1}{2}\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)$?

(b) Find a general formula for $d(O,a)$ where a is Euclidean distance r from the origin. What happens when r is close to 0? When r is close to 1?

(c) If x is a real number, the hyperbolic tangent function is defined by

$$\tanh(x) = \frac{1-e^{-2x}}{1+e^{-2x}}$$

Write the formula from (e) in terms of the inverse hyperbolic tangent $\tanh^{-1}$.

Problem 2.6. (a) Sketch some hyperbolic triangles.

(b) Show that the sum of the angles of a hyperbolic triangle is less than π radians.

(c) What happens in the limiting case when the vertices of the triangle are ideal points?

(d) Prove that the sum of the angles of a hyperbolic quadrilateral is less than 2π radians.

Problem 2.7 (Transitivity)**.**

For any two points $z, w \in \mathbb{D}$, prove that there is an isometry that maps z to w.

Problem 2.8 (SAS Congruence)**.** If ABC and DEF are two hyperbolic triangles with $\angle B = \angle E$, $d(A,B) = d(D,E)$, and $d(B,C) = d(E,F)$, prove that $ABC \cong DEF$ (That is, there is an isometry taking ABC to DEF).

Problem 2.9 (AAA Congruence)**.** If ABC and DEF are two hyperbolic triangles with $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$, then $ABC \cong DEF$.

Problem 2.10. A hyperbolic circle of radius r and center $z \in \mathbb{D}$ is the set of all points w such that $d(z,w) = r$.

(a) Show that a Euclidean circle centered at the origin is also a hyperbolic circle.

(b) Show that any Euclidean circle contained inside the Poincaré disc is also a hyperbolic circle. Does the center of the hyperbolic circle coincide with the center of the Euclidean circle?

Problem 2.11. (a) Which regular polygons can be used to tessellate the Euclidean plane (without gaps)?

(b) Figure 4 shows a tessellation of the hyperbolic plane by regular pentagons. Which regular polygons can be used to tessellate the Poincaré disc?

Definition 2.12. Given a line ℓ and a point P not on ℓ, there are rays r_1 and r_2 through P that are parallel to ℓ and such that any ray in the interior of $\angle(r_1, r_2)$ intersects ℓ. Drop a perpendicular from P to ℓ. The *angle of parallelism* is the angle between the perpendicular and r_1 (or equally, r_2).

Problem 2.13 (Lobachevskii's Theorem)**.** The angle of parallelism θ associated to P is related to the hyperbolic length d of the perpendicular by

$$e^{-d} = \tan\frac{\theta}{2}$$

Hint: Transform ℓ so that it is a diameter and $\ell \perp OP$. Draw the tangent to r_1 at P and look at the triangle formed with the endpoint of ℓ.

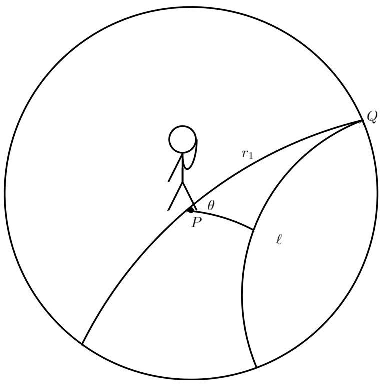

Figure 5: The angle of parallelism θ: If a person living in the Poincaré disc was standing at a point P, that person can find out the distance to the line ℓ just by looking down the ray r_1 parallel to ℓ, calculating the angle θ, then using Lobachevskii's theorem. Strange!

Problem 2.14. For any hyperbolic triangle that is right-angled and isosceles, there is an upper bound for the hyperbolic length of its altitude. What is the value of this upper bound? This is called *Schweikart's Constant.*

Problem 2.15 (Pythagorean Theorem)**.** The hyperbolic sine and cosine functions are defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh = \frac{e^x + e^{-x}}{2}$$

(a) Show $\tanh(x) = \sinh(x)/\cosh(x)$.

(b) Show that $\cosh^2(x) - \sinh^2(x) = 1$.

(c) Let ABC be a hyperbolic triangle with $\angle C = \pi/2$. Prove the following identities:

$$\sin\angle A = \frac{\sinh a}{\sinh c}, \quad \cos\angle A = \frac{\tanh a}{\tanh c}$$

where $c = d(A, B)$, $a = d(C, B)$, and $b = d(C, A)$.

(d) Prove the hyperbolic Pythagorean theorem:

$$\cosh a \cosh b = \cosh c.$$

Problem 2.16. (a) If $a \in \mathbb{D}$, show that the complex function

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a}z}$$

is an isometry that maps a to 0. ($\bar{a}$ is the complex conjugate of a).

(b) Is this the same isometry that was constructed in Problem 1.4? Prove it.