Homework 5: Quadratic Inequalities

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1 Reading

Solution 1 (L3.4)**.**
Suppose our quadratic equation is $ax^2 + bx + c = 0$ and has roots x_0, x_1. Then by Vieta's Theorem we know that $b = -a(x_0 + x_1)$ and $c = a(x_0 x_1)$. Since all the number involved are integers, we know that $a \mid c$, $x_0 \mid c$ and $x_1 \mid c$. So c is divisible by at least three of the other numbers. But among the four integers left on the board there is only one pair where one is divisible by another: $2 \mid 4$. This means that the erased number must have been c. Since Vieta's theorem also tells us that $a \mid b$, we must have $a = 2$ and $b = 4$. Then the roots are $3, -5$, and we can find c via $c = a(x_0 x_1) = 2 \cdot 3 \cdot (-5) = -30$, which is the answer

Solution 2 (L3.2b)**.**
Suppose we want to write our function f as $g + h$, where g is even and h is odd. This means that for any $x \in \mathbb{R}$ we have

$$g(-x) = g(x)$$

$$h(-x) = -h(x)$$

$$f(x) = g(x) + h(x)$$

We also know that

$$f(-x) = g(-x) + h(-x) = g(x) - h(x)$$

Adding the last two equations and subtracting we get

$$f(x) + f(-x) = 2g(x)$$

$$f(x) - f(-x) = 2h(x)$$

or, after dividing by 2,

$$\frac{f(x) + f(-x)}{2} = g(x)$$

$$\frac{f(x) - f(-x)}{2} = h(x)$$

This takes care of the uniqueness part – we just showed that if such g and h do exist, they must be given exactly by formulas

$$g(x) = \frac{f(x) + f(-x)}{2}$$

$$h(x) = \frac{f(x) - f(-x)}{2}$$

for all real x. On the other hand, this also gives us the existence. If we define g and h as above, then they are indeed even and odd:

$$g(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = g(x)$$

$$h(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -h(x)$$

And for all $x \in \mathbb{R}$ we have

$$g(x) + h(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x)$$

so we are done.

2 Homework

Problem 1.
Without computing the roots x_0, x_1 of the equation $3x^2 + 8x - 1$, determine the following quantities:

a) $x_0 x_1^4 + x_1 x_0^4$

b) $x_0^4 + x_1^4$

c) $x_0/x_1 + x_1/x_0$

Problem 2.
The quadrilateral $ABCD$ is inscribed and circumscribed at the same time, and the centers of its incircle and circumcircle coincide. Show that $ABCD$ is a square.