

# Egyptian Fractions II

Recall that the Egyptians represented fractions using a **sum of distinct unit fractions**—in other words, each fraction has numerator 1 and all the denominators are different.

For example,

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4} \quad \text{or} \quad \frac{6}{7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{42}.$$

(But not  $2/5 = 1/5 + 1/5$ , because the unit fractions must be different from each other.)

It is always possible to find an Egyptian representation for a fraction  $p/q < 1$  using a method called the *Greedy Algorithm*. Here's an example of how the Greedy Algorithm works:

The steps of the Greedy Algorithm for a fraction  $p/q$  are:

1. Find the biggest unit fraction smaller than  $p/q$ —suppose it's  $1/k$ . Then add  $1/k$  to the list of unit fractions which will add up to  $p/q$ . **[For example, if we want to find an Egyptian representation for  $2/5$ , we would choose  $1/k$  to be  $1/3$  because it's the biggest unit fraction smaller than  $2/5$  ( $1/2$  is too big!). So  $2/5 = 1/3 + \dots$ ]**
2. Compute, in lowest terms,  $c/d = p/q - 1/k$ . If  $c/d$  is a unit fraction, add it to the list of unit fractions which add up to  $p/q$ , and you're all done! **[So  $c/d = 2/5 - 1/3 = 6/15 - 5/15 = 1/15$ . Since this is a unit fraction, we add it to our list, and we're done!  $2/5 = 1/3 + 1/15$ .]**
3. If  $c/d$  in Step 2 is *not* a unit fraction, then start over with step 1, but now use  $c/d$  in place of  $p/q$ .

**Another Example:** To find an Egyptian fraction for  $4/5$  using the Greedy Algorithm:

1. Find the largest unit fraction smaller than  $4/5$ . This turns out to be  $1/2$ . Add  $1/2$  to the list of unit fractions. ( $4/5 = 1/2 + \dots$ )
2. Compute  $c/d = 4/5 - 1/2 = 8/10 - 5/10 = 3/10$ .
3. Since  $3/10$  is not a unit fraction, we start over using  $3/10$  as the fraction.
  1. The largest unit fraction smaller than  $3/10$  is  $1/4$ . Add  $1/4$  to the list of unit fractions. ( $4/5 = 1/2 + 1/4 + \dots$ )
  2. Compute  $c/d = 3/10 - 1/4 = 6/20 - 5/20 = 1/20$ . Since this is a unit fraction, add  $1/20$  to the list of unit fractions and we are done! We found that  $4/5 = 1/2 + 1/4 + 1/20$ .

1. Use the Greedy Algorithm to find an Egyptian representation for:

(a)  $10/21$

(b)  $5/21$

(c)  $13/42$

(d)  $5/91$

2. In parts (b), (c), and (d) of the previous problem, the Greedy Algorithm gives an answer with three or more unit fractions. Can you find ways to represent the same original fraction using only *two* unit fractions?

(b)  $5/21$

(c)  $13/42$

(d)  $5/91$

3. Use the Greedy Algorithm to find Egyptian representations for:

(a)  $2/3$

(b)  $2/5$

(c)  $2/7$

(d)  $2/9$

(e)  $2/11$

(f)  $2/13$

(g)  $2/(2k-1)$ , for  $k \geq 2$