INFINITE SETS

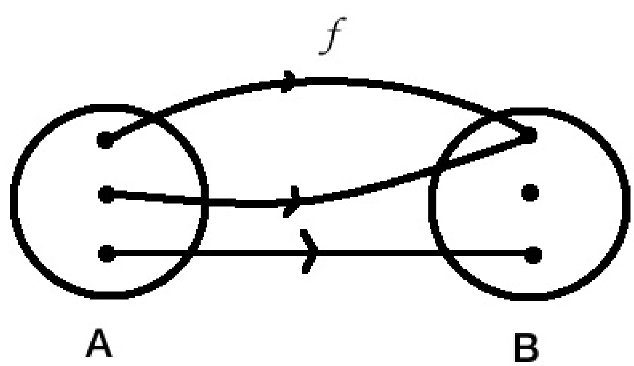
BEGINNERS 10/26/2014

We have seen that

- If two finite sets A and B have the same number of elements, there is a function $f : A \to B$ that is both one-to-one and onto.
- If there is a function $f : A \to B$ that is both one-to-one and onto, the number of elements in A and B is the same.

We will call two such finite sets *equivalent*.

(1) Consider a function $f : A \to B$ defined by the diagram below



Is this function defined?

Is it onto?

Is it one-to-one?

Can you conclude that sets A and B have different numbers of elements?

(2) Construct a function from the set $A = \{1, 2, 3, 4, 5\}$ to the set $B = \{a, b, c, d, e\}$ which is not onto and/or not one-to-one.

Can you conclude that A and B are not equivalent?

(3) How can we compare the size of two *infinite* sets using functions?

Definition 1. Two sets A and B (finite or infinite) are *equivalent* if there is a function $f : A \to B$ such that

- f is one-to-one;
- f is onto;

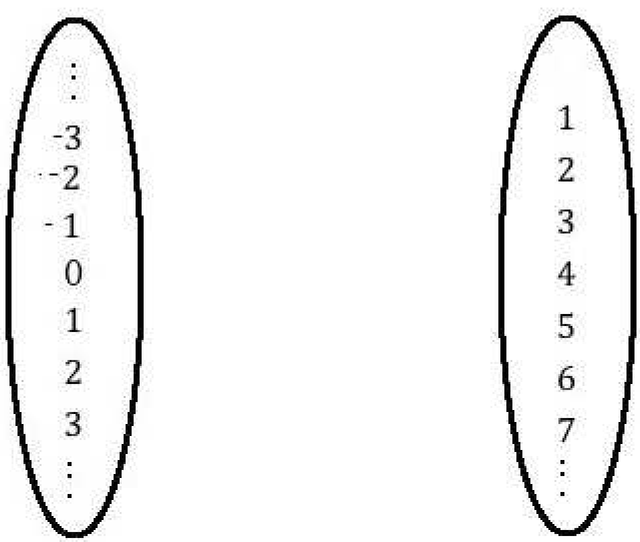
Note that if we want to show that A and B are not equivalent, we have to prove that no such function exists.

Definition 2. A set A is called *countable* if it is equivalent to the set $\mathbb{N} = \{1, 2, 3, 4, 5...\}$.

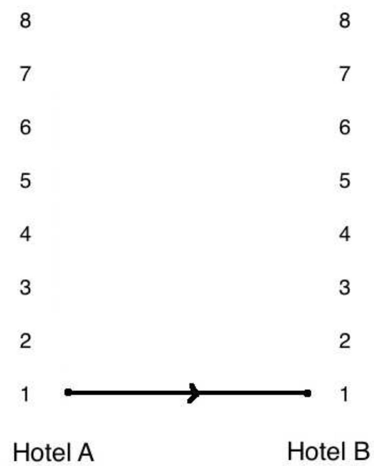
(1) Explain the name *countable*.

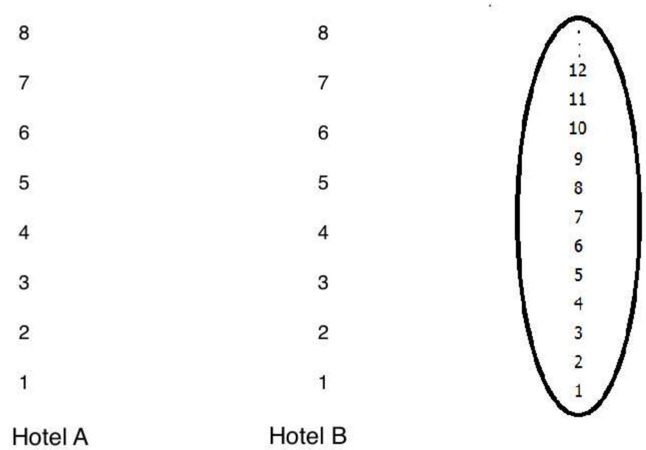
(2) Enumerate elements of $\mathbb{Z} = \{... - 3, -2, -1, 0, 1, 2, 3...\}$ starting in the following way $0 \mapsto 1 \mapsto -1 \ldots$.

Are the sets $\mathbb{Z}$ and $\mathbb{N}$ equivalent? Does a function $f : \mathbb{Z} \to \mathbb{N}$ which is both onto and one-to-one exist? Can you define it?



(3) Consider the set of rooms in two Hotels *Infinity*. Can you enumerate the elements of this set starting with Room 1 in Hotel A and continuing to Room 1 in Hotel B?





(a) Does this define a function from the set of rooms in the hotels to$\mathbb{N}$?

(b) Is this function onto?

(c) Is this function one-to-one?

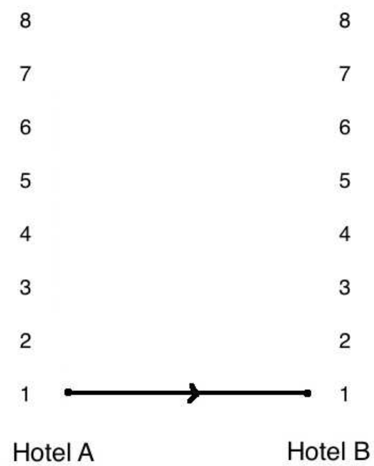
(d) Is the set of rooms in two hotels infinitely countable?

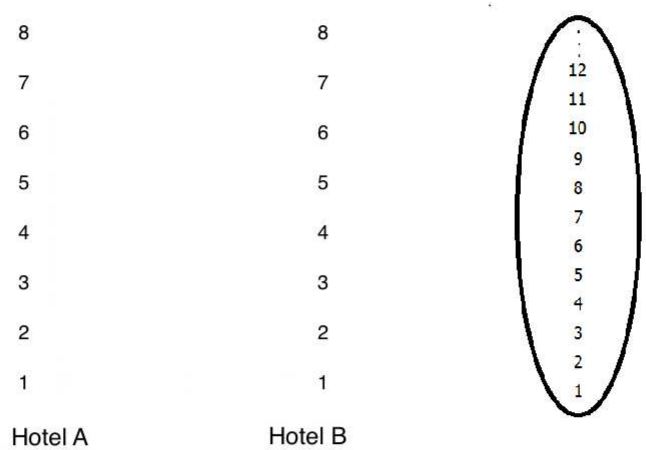
(e) Let $n_A$be the room on the n^{th} floor of Hotel A, and let n_B be the room on the n^{th} floor of Hotel B. Write down the values for the function above for n_A and n_B.

$f(n_A) =$

$f(n_B) =$

(4) Can you find another way to enumerate the elements of the set of rooms in two Hotels *Infinity* starting with Room 1 in Hotel A and continuing to Room 1 in Hotel B?





(a) Does this define a function from the set of rooms in the hotels to$\mathbb{N}$?

(b) Is this function onto?

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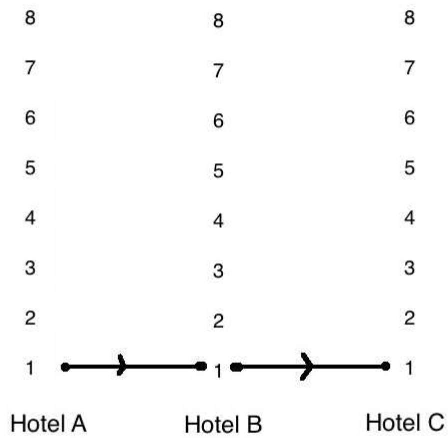
(d) Is the set of rooms in two hotels infinitely countable?

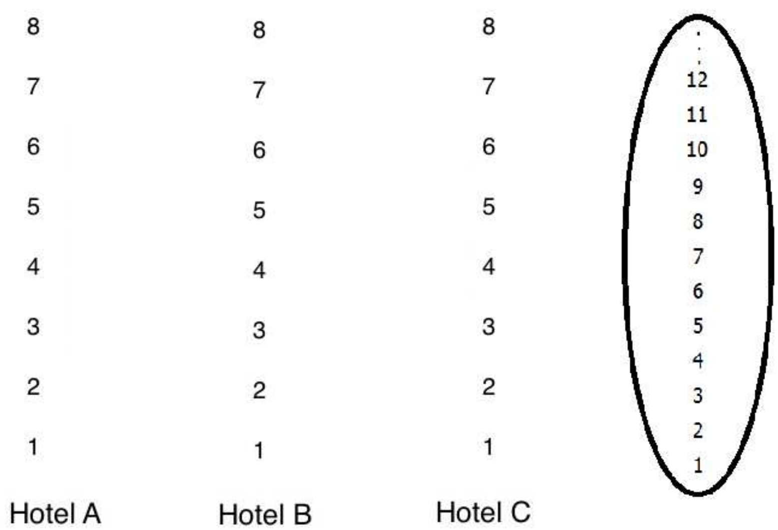
(e) Let $n_A$be the room on the n^{th} floor of Hotel A, and let n_B be the room on the n^{th} floor of Hotel B. Write down the values for the function above for n_A and n_B.

$f(n_A) =$

$f(n_B) =$

(5) Consider the set of rooms in three Infinity Hotels. Can you enumerate the elements of this set?





(a) Does this define a function from the set of rooms in the hotels to ℕ ?

(b) Is this function onto?

(c) Is this function one-to-one?

(d) Is the set of rooms in two hotels infinitely countable?

(e) Let n_A designate the room on the n^{th} floor of Hotel A, n_B the room on the n^{th} floor of Hotel B, and n_C the room on the n^{th} of Hotel C.Write down the values for the function above for n_A, n_B:and n_C.

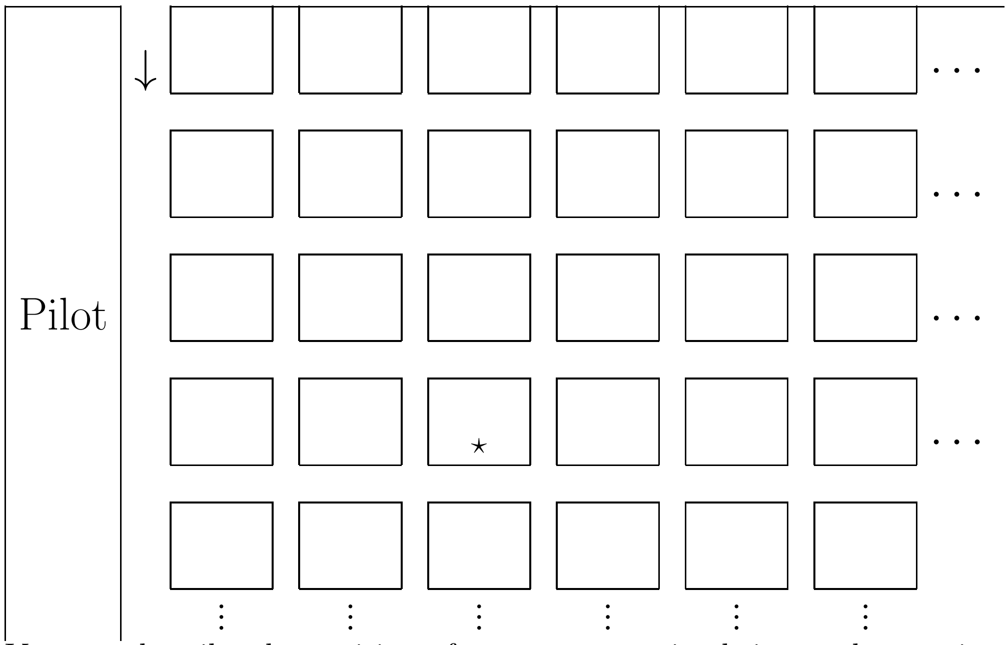
$f(n_A) =$

$f(n_B) =$

$f(n_C) =$

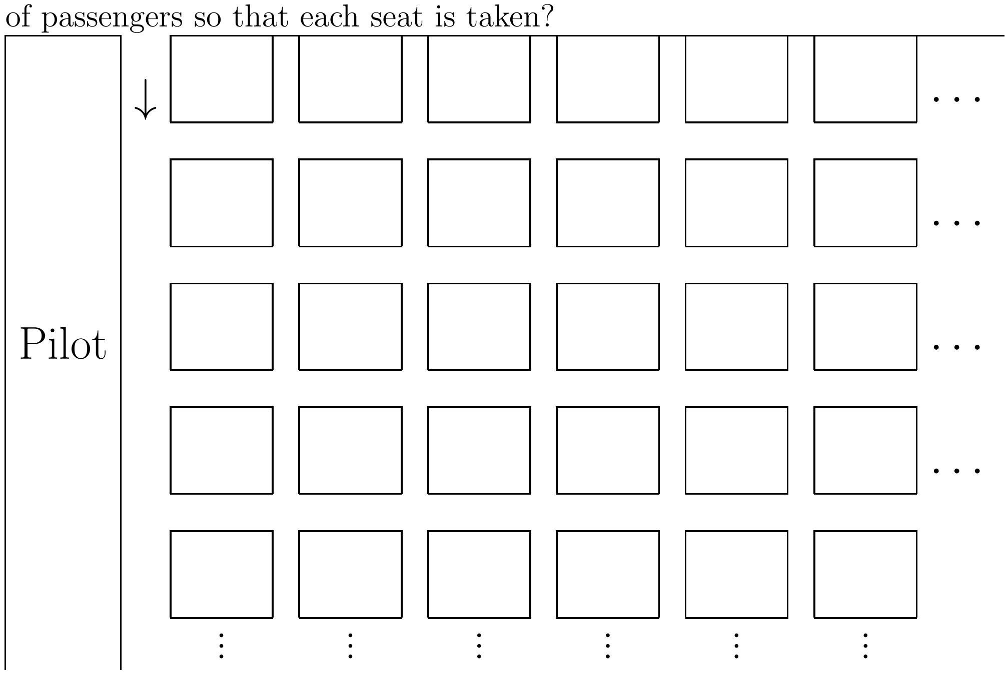
(f) Does this mean that 3 Infinity Hotels are equivalent to 1 Hotel Infinity?

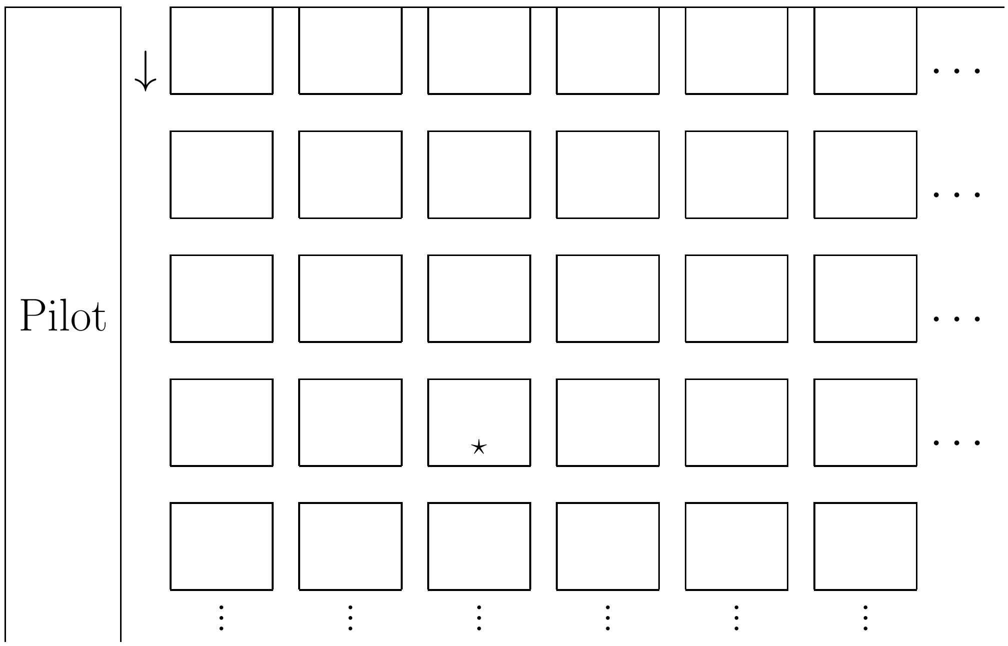
(6) A super-wide-body rocket has an infinite number of seats in each row. Can you assign seats to an infinite number of passengers so that each seat is taken? Draw an arrow that shows the order in which you will assign seats.



You can describe the position of any passenger in their seat by a pair of numbers. For example, the sit marked by $\star$ is described by the pair $(3, 4)$ (which means that you go 3 rows to the right and 4 rows down to get there from the entrance)

Can you come up with two more possible ways to assign seats to an infinite number of passengers so that each seat is taken?





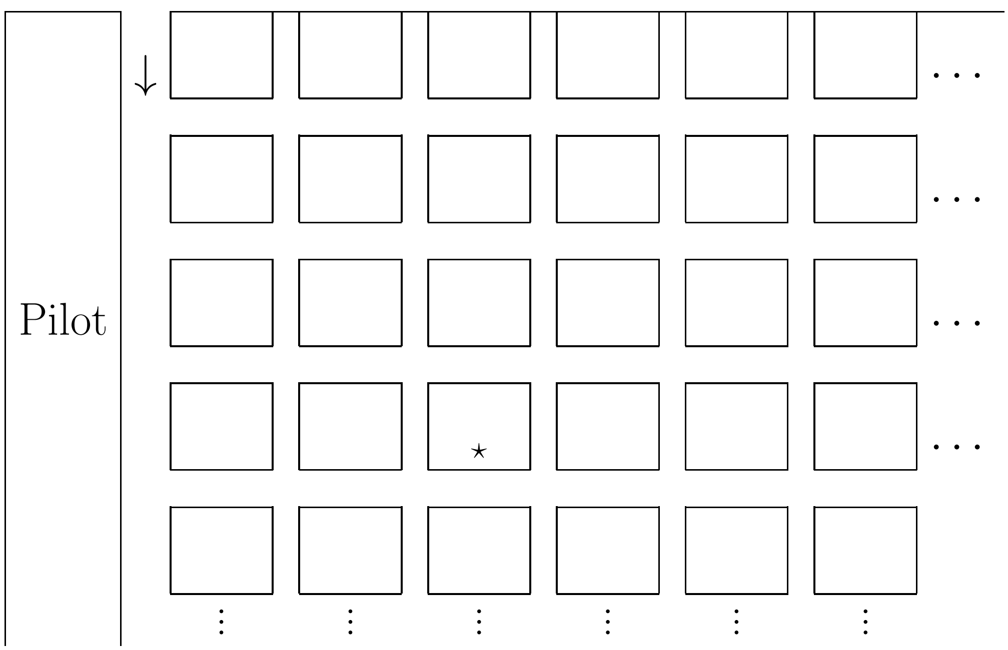
(a) Is the set consisting of pairs of natural numbers countable?

This means that infinitely many Infinity Hotels is equivalent to just one. AMAZING!

(b) In your own words, can you explain how it is that countably many Infinity Hotels are equivalent to one Infinity Hotel?

(7) Natural numbers, $\mathbb{N}$, are the numbers we use when we count. Rational numbers,$\mathbb{Q}$, are the numbers that can be expressed as fractions.
(a) Which set do you expect to have more elements and why?

(b) Consider the super-wide-body rocket which has an infinite number of seats in each row. In problem 6, we already determined that the seats in the rocket can be enumerated. Thus, the set of seats in this rocket is a countable set (i.e., equivalent to N). Let us consider the rational numbers. Can you assign all the positive rational numbers to the seats in the super-wide-body rocket? Before assigning the numbers, label the axes of the rocket with the natural numbers such that the sit marked by $\star$ can be described by the pair $(3, 4)$, and conect the seats by a path. (*Hint*: Start by labeling (1,1) by 1/1.)



(a) Are any of the fractions equivalent to each other? Only keep one of the fractions.

(b) Is the set of rational numbers countable?

(c) Revisit the question in part (a), which set,$\mathbb{N}$ or $\mathbb{Q}$, has more elements? Explain why this is surprising.

(d) Do you expect that there exists a function, $f : \mathbb{Q} \to \mathbb{N}$, that is both onto and one-to-one?

	1	2	3	4	5	...
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	...
2	$\frac{1}{2}$	~~$\frac{2}{2}$~~	$\frac{3}{2}$	~~$\frac{4}{2}$~~	$\frac{5}{2}$	...
3	$\frac{1}{3}$	$\frac{2}{3}$	~~$\frac{3}{3}$~~	$\frac{4}{3}$	$\frac{5}{3}$	...
4	$\frac{1}{4}$	~~$\frac{2}{4}$~~	$\frac{3}{4}$	~~$\frac{4}{4}$~~	$\frac{5}{4}$	...
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	~~$\frac{5}{5}$~~	...
...	...	...	...	...	...	...

8. Write the following numbers in binary notation.

Example: $19_{10} = 10011_2 = \mathbf{1}\cdot 2^4 + \mathbf{0}\cdot 2^3 + \mathbf{0}\cdot 2^2 + \mathbf{1}\cdot 2^1 + \mathbf{0}\cdot 2^0$

[The subscript 10 means the number is written in decimal notation.]

a) $8 =$

b) $21 =$

c) $48 =$

d) $1043 =$

9. Four houses A, B, C, and D are built along a straight road.

You are an engineer commissioned to find a place for a water well W so that the total distance from W to A, B, C, and D is the shortest possible. Where would you place the well?

10. Divide a clock face with two straight lines so that the sum of the numbers in each part are equal.

11. Now divide a clock face into 6 parts using straight lines in such a way that each part contains two numbers, and the six sums of the two numbers are equal.

12. (Canadian Math Kangaroo) There are between 50 and 100 students in Beginner's Math Circle. Cory tried to group the students in teams of 5, 6 and 12 per team, and noticed that there were always 3 students left. How many students are in the Beginner's Math Circle?

13. (Bulgarian Math Olympiad) A boy and a pig weigh as much as 5 sheep. Two cats and a pig weigh as much as 3 sheep, and one pig weighs as much as 4 cats. As much as how many cats does one boy weigh?

14. (Bulgarian Math Olympiad) To feed a total of 56 dogs and cats we need 304 sausages. Every cat eats 5 sausages and every dog eats 6. How many are the cats?