COMPLEX NUMBERS AND PLANE GEOMETRY
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Some polynomial equations (for example, $x^2+1=0$) do not have solutions in the real numbers. Complex numbers were introduced in order to solve such equations. If we denote $i=\sqrt{-1}$, then any complex number is of the form $z=a+bi$, where a and b are real numbers. Complex numbers can be added and multiplied:

$$(a+bi)+(c+di)=(a+b)+(c+d)i,$$
$$(a+bi)(c+di)=(ac-bd)+(ac+bd)i.$$

From now on, we will see any complex number $z=a+bi$ as a point in the plane whose coordinates are a and b. This will enable us to solve geometry problems by using complex numbers. The *absolute value* $|z|$ of z is by definition the distance between 0 and z:

$$|z|=\sqrt{a^2+b^2}$$

The *argument* $\arg z$ of a non-zero complex number z is the counterclockwise angle between the line $0z$ (going through 0 and z) and the x-axis. Then z can be written as

$$z=|z|\,(\cos(\arg z)+\sin(\arg z)\,i)$$

The *complex conjugate* of $z=a+bi$ is defined as $\overline{z}=a-bi$. Using the complex conjugate we can divide by any non-zero complex number z by following the rule $\frac{w}{z}=\frac{w\overline{z}}{|z|^2}$.

Problem 1. Let z,w be complex numbers.

(a) Prove that the distance between z and w is equal to $|z-w|$.

(b) Prove that $|z+w|\leq|z|+|w|$.

(c) Prove that $0,z,z+w,w$ form a parallelogram.

Problem 2. Show that if z,w are complex numbers, then

$$|zw|=|z||w|,\ \ \arg zw=\arg z+\arg w \text{ (modulo } 360^o).$$

Problem 3. Let ABCD be a quadrilateral. Prove that

$$\text{AB}\cdot\text{CD} + \text{AD}\cdot\text{BC} \geq \text{AC}\cdot\text{BC}.$$

Problem 4. Let z_1,z_2,z_3,z_4 be four distinct complex numbers. Prove that the lines z_1z_2 and z_3z_4 are perpendicular if and only if $\frac{z_1-z_2}{z_3-z_4}$ is an imaginary number (that is, of the form bi, for some real number b).

Problem 5. Construct two squares ABXY and ACZT in the exterior of a triangle ABC. Prove that the midpoint of the segment XZ is independent of A.

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Problem 6. Prove that four distinct points z_1, z_2, z_3, z_4 in the plane lie on a circle if an only if $\frac{z_1-z_3}{z_2-z_3} / \frac{z_1-z_4}{z_2-z_4}$ is a real number.

Problem 7 (Ptolemy's theorem). Show that if four points A,B,C,D lie on a circle, in this order, then

$$\mathrm{AB}\cdot\mathrm{CD} + \mathrm{AD}\cdot\mathrm{BC} = \mathrm{AC}\cdot\mathrm{BC}.$$

Problem 8. Prove that three distinct points z_1, z_2, z_3 in the plane form an equilateral triangle if and only if $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$.

Problem 9. Let ABC be a triangle. On the sides AB,BC,CA consider points X,Y,Z which divide the sides into the same ratio. Prove that the triangle XYZ is equilateral if and only if the triangle ABC is equilateral.

Problem 10 (Euler's line). The circumcentre O, the centre of gravity (centroid) G and the orthocentre H of a triangle lie on the same line, in this order. Moreveor, we have that OH = 3 OG.

Problem 11 (Simson's line). If A,B,C are points on a circle, then the feet of perpendiculars from an arbitrary point D on that circle to the sides of ABC lie on a line.

Problem 12. Let M and N be interior points of the triangle ABC such that $\widehat{\mathrm{MAB}} = \widehat{\mathrm{NAC}}$ and $\widehat{\mathrm{MBA}} = \widehat{\mathrm{NBC}}$. Prove that

$$\frac{\mathrm{AM}\cdot\mathrm{AN}}{\mathrm{AB}\cdot\mathrm{AC}} + \frac{\mathrm{BM}\cdot\mathrm{BN}}{\mathrm{BA}\cdot\mathrm{BC}} + \frac{\mathrm{CM}\cdot\mathrm{CN}}{\mathrm{CA}\cdot\mathrm{CB}} = 1.$$

Problem 13. On each side of a triangle construct equilateral triangles, lying exterior to the original triangle. Show that the centroids of the three equilateral triangles form themselves an equilateral triangle.

Problem 14. Let $\mathrm{A}_1\mathrm{A}_2 \ldots \mathrm{A}_n$ be a regular n-gon inscribed in a unit circle. Prove that the product of the distances from A_1 to all of the other $(n-1)$ vertices is equal to n:

$$\mathrm{A}_1\mathrm{A}_2\cdot\mathrm{A}_1\mathrm{A}_3\cdot\ldots\ \cdot\mathrm{A}_1\mathrm{A}_n = n.$$