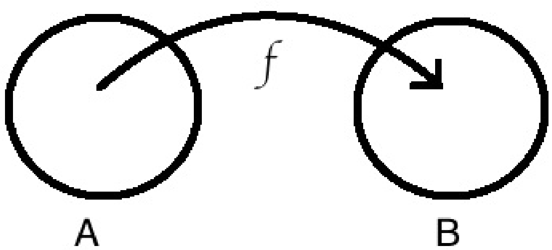
FUNCTIONS ON SETS

BEGINNERS 10/19/2014

A *function* from a set A to a set B is a map or a rule that assigns to each element in A an element in B. The rule can be expressed in words, as a picture, by a table of values or as by formula.



Example 1. Let S be the set of all students in the Beginners group (who are in class today). Let C be the set of all chairs in the Beginners classroom. Define a function from the set of students S to the set of chairs C that assigns to each student the chair on which he or she is sitting.

(a) Give two examples of elements of S.

(b) Do we know the value of this function for every student? If not, give an example when the function is not defined.

Example 2. Let S be the set of all students in the Beginners group (who are in class today). Let I be the set of instructors in the Beginners group. Define a function $f : S \to I$ from the set of students S to the set of instructors I that assigns to each student the instructor who sits at the same table.

The value of the function f on an element x from S is denoted by $f(x)$.

(a) List the elements of I.

I={__________________, __________________, __________________, __________________}

(b) Compute the function for the following students:

$f(Elijah) =$

$f(Jocelyn) =$

$f(Mia) =$

$f(Lola) =$

(b) Is there a student such that the value of the function for this student equals EMMANUELLE? Explain.

(c) Is $f(Lucy)$ defined? Explain.

Example 3. Let $A = \{0, 1, 2, 3, 4, 5\}$ be a set. Define a function $f : A \to A$ by:

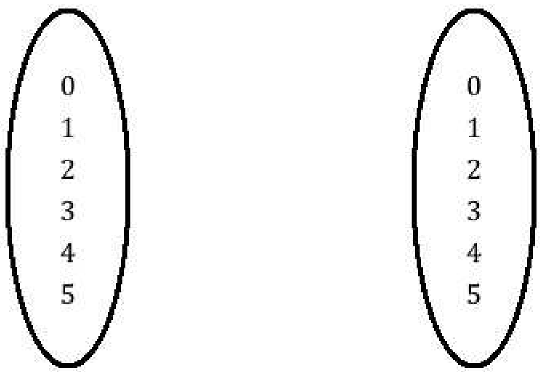
$$f(x) = 5 - x$$

That is, for every number x, the function converts it to $5 - x$.

Make a table of values for this function:

x	0	1	2	3	4	5
f(x)						

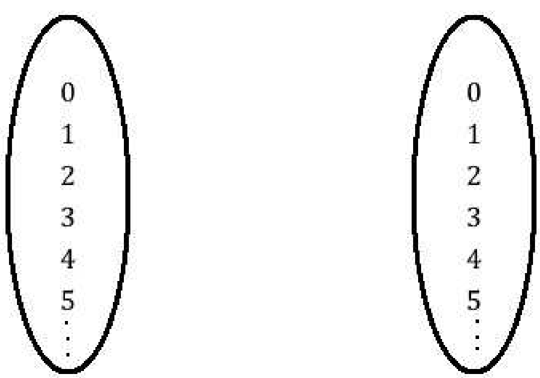
Draw an arrow from each number on the left to the corresponding number on the right:



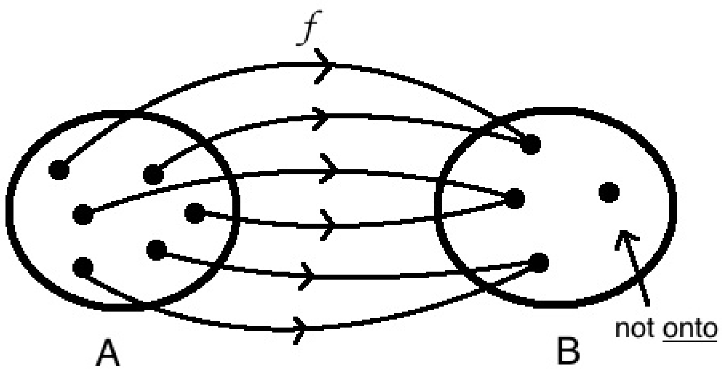
Example 4. Let $\mathbb{N} = \{1, 2, 3, 4...\}$ be the set of natural numbers (these are the number we use when we count). Let $f : \mathbb{N} \to \mathbb{N}$ be a function that takes any natural number to 3. Write down a formula describing this function

$$f(x) =$$

Draw an arrow from each number on the left to the corresponding number on the right:



Definition 5. A function $f : A \to B$ from a set A to a set B is called *onto* if all the elements in B can be obtained as a result of the function. .



Go back to Examples $1-4$ and label each function as "onto" or "not onto".

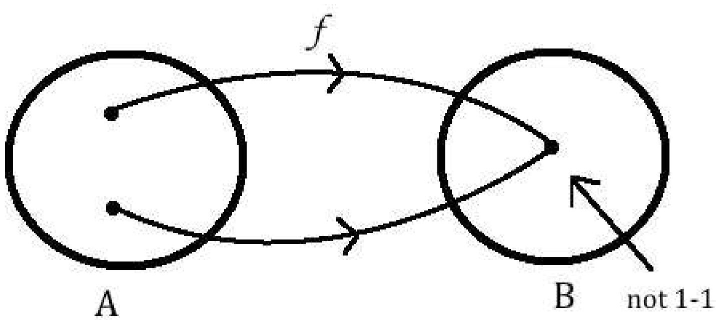
(1) $S \to C$ Onto: Yes / No

(2) $S \to I$ Onto: Yes / No

(3) $f(x) = 5 - x$ for $x\backslash in\{0, 1, 2, 3, 4, 5\}$ Onto: Yes / No

(4) $f(x) = 3$ Onto: Yes / No

Definition 6. A function $f : A \to B$ from a set A to a set B is called $one-to-one$ if each of the values in B comes from at most 1 in A.



Go back to Examples $1-4$ and label each function as "one-to-one" or "not one-to-one".

(1) $S \to C$ One-to-one: Yes / No

(2) $S \to I$ One-to-one: Yes / No

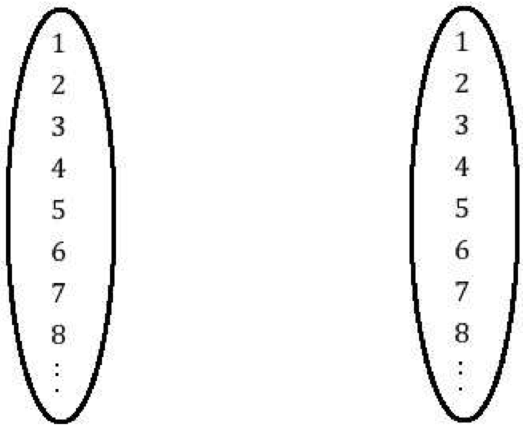
(3) $f(x) = 5 - x$ for$x\backslash in\{0, 1, 2, 3, 4, 5\}$ One-to-one: Yes / No

(4) $f(x) = 3$ One-to-one: Yes / No

(1) All functions below take a natural number and produce a natural number.

(a) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x + 3$

(i) Draw an arrow from each number on the left to the corresponding number on the right:



(ii) Can you get all numbers in $\mathbb{N}$ as a result of the function? If not, what are the numbers that you can not get?

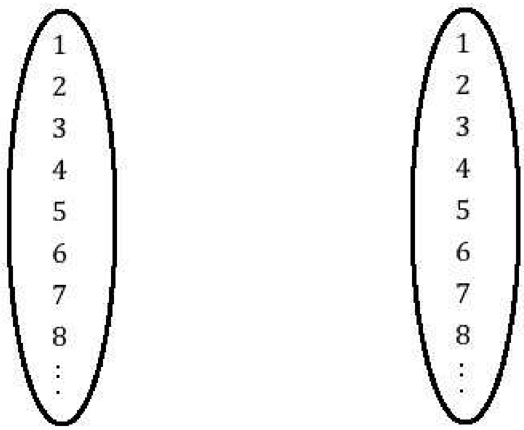
(iii) Is this function onto?

(iv) Is it true that any value can be obtained in *only* 1 way? If not, what values have the same output?

(v) Is this function one-to-one?

(b) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ 2 & \text{if } x \text{ is odd} \end{cases}$

(i) Draw an arrow from each number on the left to the corresponding number on the right:



(ii) Can you get all numbers in $\mathbb{N}$ as a result of the function? If not, what are the numbers that you can not get?

(iii) Is this function onto?

(iv) Is it true that any value can be obtained in *only* 1 way? If not, what values have the same output?

(v) Is this function one-to-one?

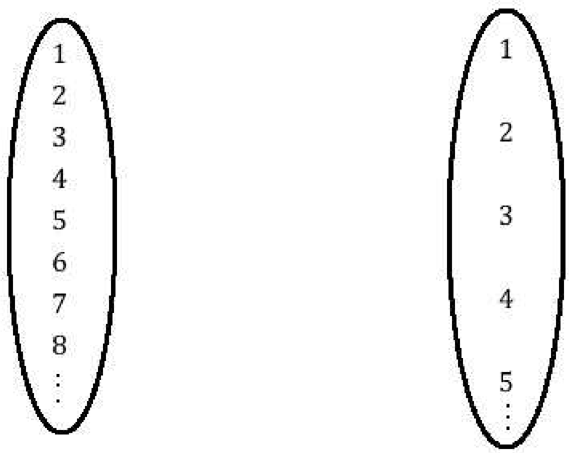
(c) $f(x)$ is such that:

x	1	2	3	4	5	6	7	8	⋯
f(x)	1	1	2	2	3	3	4	4	⋯

(i) What happens if x is even? Describe the function in words.

(ii) What happens if x is odd? Describe the function in words.

(iii) Draw an arrow from each number on the left to the corresponding number on the right:



(iv) Can you get all numbers in $\mathbb{N}$ as a result of the function? If not, what are the numbers that you can not get?

(v) Is this function onto?

(vi) Is it true that any value can be obtained in *only* 1 way? If now, what values have the same output?

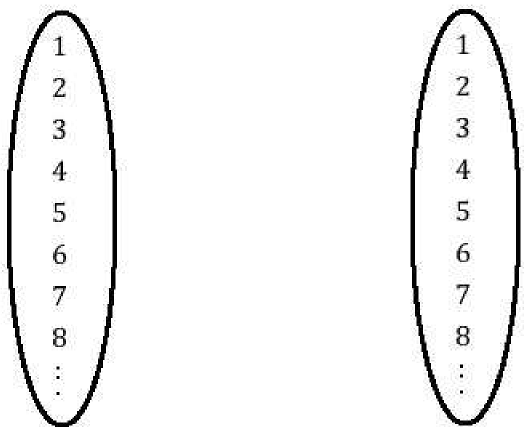
(vii) Is this function one-to-one?

(d) $f(x)$ is such that:

x	1	2	3	4	5	6	7	8	$\cdots$
f(x)	2	1	4	3	6	5	8	7	$\cdots$

(i) Describe the function in words.

(ii) Draw an arrow from each number on the left to the corresponding number on the right:



(iii) Can you get all numbers in $\mathbb{N}$ as a result of the function? If not, what are the numbers that you can not get?

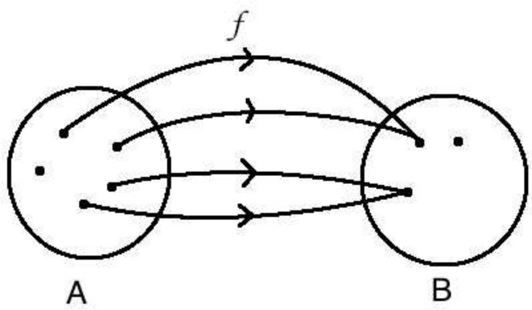
(iv) Is this function onto?

(v) Is it true that any value can be obtained in *only* 1 way? If now, what values have the same output?

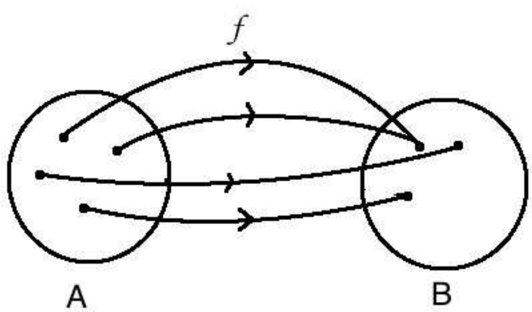
(vi) Is this function one-to-one?

(2) Use the pictures to answer the questions below.

(a) Determine whether this function is defined on A.

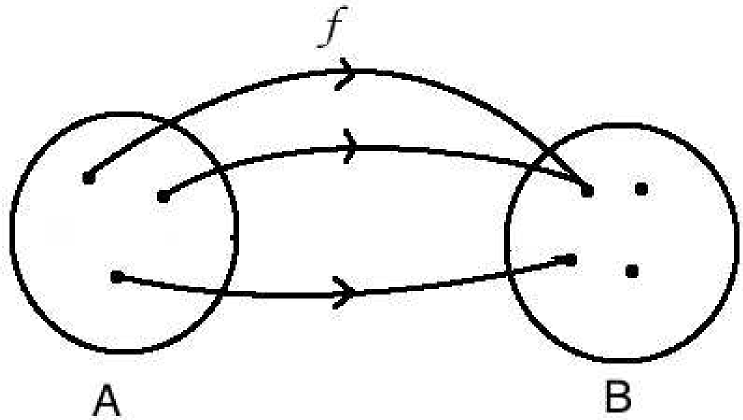


(b) Determine whether this function is defined on A.



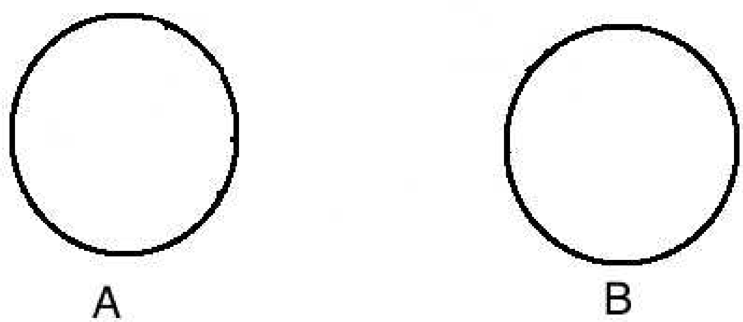
Onto: Yes/ No
One-to-one: Yes/ No
Provide an explanation for your choices. Circle the relevant parts of the diagrams using two different colors, one for onto and one for one-to-one, as necessary.

(c) Determine whether this function is defined on A.



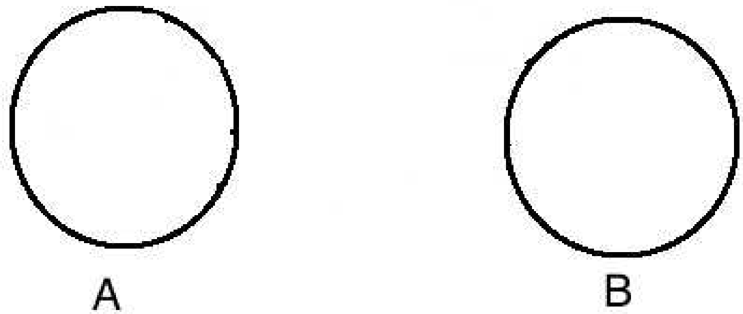
Onto: Yes/ No
One-to-one: Yes/ No
Provide an explanation for your choices.

(d) Come up with your own example of a function that is onto, but not one-to-one. Circle the relevant parts of the diagrams using two different colors, one for onto and one for one-to-one, as necessary.



Which set has more elements?

(e) Come up with your own example of a function that is one-to-one and onto.

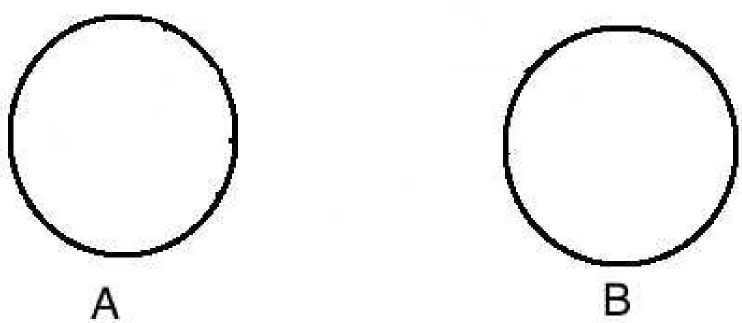


Which set has more elements?

(3) Let A and B be two sets each consisting of n elements (where n is a finite number). Explain how we can construct a function $f : A \to B$ such that

- f is onto and
- f is one-to-one

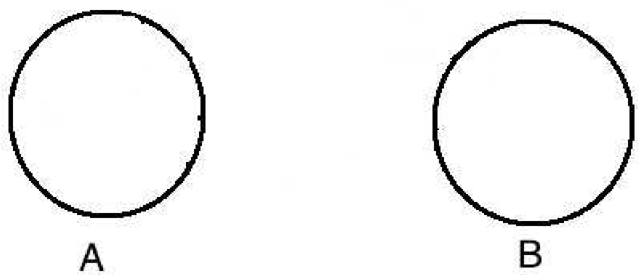
(*Hint*: number the elements of each set first. Come up with your own picture.)



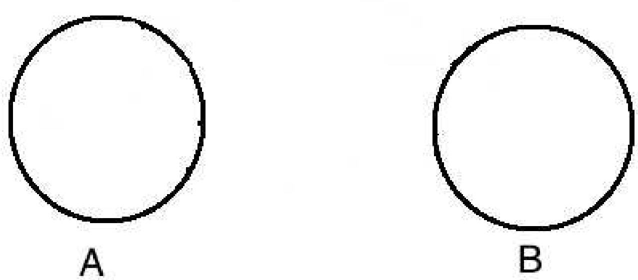
(4) Suppose that A and B are sets such that there is a function $f : A \to B$ which is onto and one-to-one.

(a) Suppose that a is the number of elements in the set A and b is the number of elements in the set B. Suppose that $a < b$.

(i) Let $f : A \to B$ be a function. Can f be onto? Make a picture in the case that a=3 and b=5.

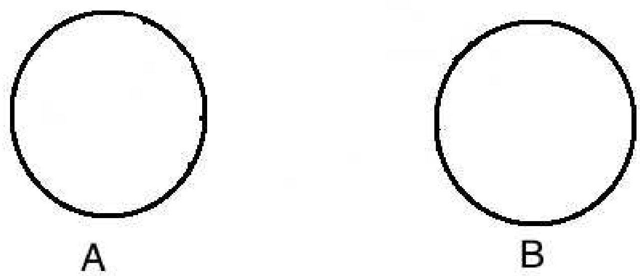


(ii) Can you always construct a function $f : A \to B$ which is one-to-one? How? Make a picture in the case that a=3 and b=5.

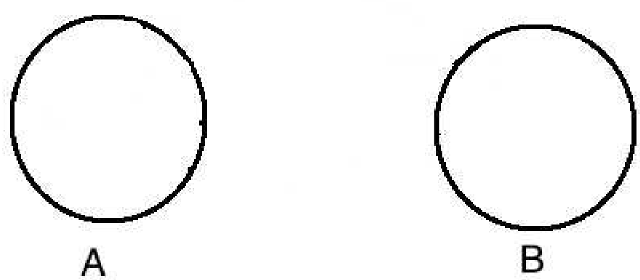


(5) Suppose that a is the number of elements in the set A and b is the number of elements in the set B. Suppose that $a > b$.

(a) Let $f : A \to B$ be a function. Can f be onto? Make a picture in the case that a=5 and b=3.



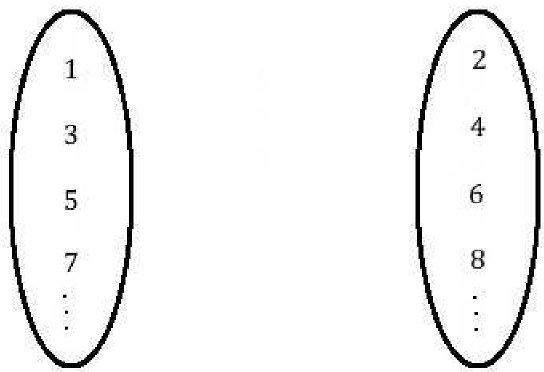
(b) Can you always construct a function $f : A \to B$ which is one-to-one? How? Make a picture in the case that a=5 and b=3.



(6) How do we compare infinite sets if we can not count the total number of elements in them? What ideas can we use?

(7) Define a function from the set of odd numbers to the set of even numbers that is

- one-to-one
- onto

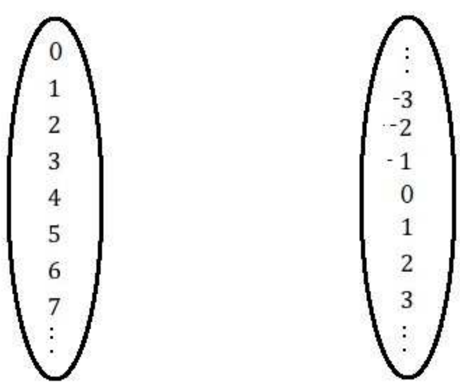


Make a conclusion. How many such functions are there?

(8) Define a function from the set of integers $\mathbb{Z} = \{... - 3, -2, -1, 0, 1, 2, 3...\}$ to the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4...\}$ that is

- one-to-one
- onto

Make a conclusion.



(9) Put the right sign, $>$, $<$, or $=$ between the fractions:

$$\frac{2013}{2014} \quad \frac{2014}{2015}$$

(10) Three houses A, B, and C are built along a straight road.

You are an engineer commissioned to find a place for a water well W so that the total distance from W to A, B, and C is the shortest possible. Where would you place the well?

(11) Use four fours to make fifty. (Note you can make any number from zero to fifty using four fours. Try it at home!)