UCLA Match Circle

October 21, 2018

A *group* is a set G with a multiplication satisfying a few rules:

1. Multiplication is associative, so $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
2. There is a member of the set called the *identity element*, which we write as e, so that for every member g of the group, $e \cdot g = g \cdot e = g$.
3. Every member g of the group has an inverse, which we call g^{-1}, which has the property that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

The set of symmetries of any object form a group. The multiplication is composition, or first doing one symmetry and then doing the next. The identity element is the symmetry that does nothing, i.e. the one that leaves the object alone. The inverse of a symmetry is just the symmetry that undoes the previous one. In the example of the square card, the inverse of the symmetry of rotation by 90 degrees clockwise is rotation by 90 degrees counter clockwise.

Remember a group does not have to have the property that $x \cdot y = y \cdot x$. We call groups that have this property commutative or abelian, and groups that don't have this property noncommutative or nonabelian. (The term abelian comes from the Norwegian mathematician Niels Henrik Abel).

We'll write g^n for $g \cdot g \cdot \ldots \cdot g$ (we've multiplied n times), and xy for $x \cdot y$.

A *subgroup* of a group is a subcollection of members of the group that is itself a group with the same multiplication rule, in other words it is closed under multiplication and taking inverses.

One really important group is the symmetric group on n-letters. We call this S_n. This is the symmetry group of a set with n-elements. We call the elements of S_n permutations.

Here is a way of writing and element of S_5. This is the permutation that sends $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 2$, $4 \mapsto 5$, and $5 \mapsto 4$.

1	2	3	4	5
3	1	2	5	4

It is enough to just write the second line, so we can write this permutation as 31254.

1. Find all subgroups of the groups we encountered last week (the Klein four group, Cyclic group of order 8, the quaternion group of order 8).
2. In the mulitplication table for a group, show that every row and every column contains every element of the group.
3. What is the inverse of the permutation 31254?

4. Which permutations have the property that they are their own inverse?
5. Develop a notion of what it means for two groups to be "the same".
6. Determine the symmetry group of a square, where you are allowed to rotate the square and flip it over. Have you seen this group before? This group is called the "dihedral group of order 8".
7. Consider the following system: I have a nickel and a quarter on a table, with the nickel to the left of the quarter. I can turn the coins over if I like, but I can't move them away from their spot on the table. What is the symmetry group of this system? The symmetries are things like "flip the left coin", "flip the right coin", etc. Have you seen it before?
8. Consider the same system as in the previous question, but now you can swap the places of the coins. My symmetries would be actions like "swap the two coins", "flip the left hand coin", "swap the coins and then flip the left one" etc. What is the symmetry group of this system? Is it abelian? Have you seen it before?
9. A set S of elements in a group *generates* the group if every element in the group can be obtained by repeated multiplications of elements in S. Find a set of permutations that generates S_n that is as small as possible.
10. An *action* of a group G on a set S is a function $f : G \times S \to S$ so that $f(gh, s) = f(g, f(h, s))$ for every $g, h \in G$ and $s \in S$, and so that $f(e, s) = s$ for every $s \in S$.
    (a) If $s \in S$, the stabilizer of s is the set of elements in G so that $f(g, s) = s$. We call this G_s. Show that G_s is a subgroup.
    (b) The symmetric group S_n acts on $\{1, 2, 3, \ldots, n\}$ by permuting 1 through n. For this action, what is the stabilizer of 1 and of 2?
    (c) The dihedral group (the group of symmetries of a square when we can rotate and flip the square) acts on $\{1, 2, 3, 4\}$ by thinking of 1 through 4 as labeling the vertices of a square. For this action, what is the stabilizer of 1 and of 2?
    (d) The action of the dihedral group on $\{1, 2, 3, 4\}$ induces an action on the diagonals of the square, so on $\{\{1, 3\}, \{2, 4\}\}$. What are the stabilizers of $\{1, 3\}$ and of $\{2, 4\}$?
    (e) If s_1 and s_2 are different element of S and $f(g, s_1) = s_2$ for some $g \in G$, what is the relationship between G_{s_1} and G_{s_2}?
    (f) For $s \in S$, the orbit of s is the set of all elements $t \in S$ so that $f(g, s) = t$ for some $g \in G$. We denote the orbit of s by $\mathcal{O}_s$. Show that the size of $\mathcal{O}_s$ multiplied by the number of elements in G_s is equal to the number of elements of G, in other words: (size of $\mathcal{O}_s$)(size of G_s) = size of G.
11. For the groups we have found so far, see if you can identify them as subgroups of the symmetric group for some n.

12. For all the subgroups you have found, you may have noticed that the number of elements in the subgroup is a divisor of the number of elements in the big group. Show that this happens in general (you probably want to use the previous question).

13. Classify all groups with a prime number of elements.

14. Find an example of a group G and a number n so that n divides G, but G has no subgroup of order n.

    Give as many examples as you can of classes of groups where this does happen, i.e. where the groups have subgroups of every order dividing the order of the group.

15. Show that every finite group is a subgroup of some symmetric group.

16. If a group G acts on sets S_1 and S_2, develop a notion of what it means for the actions to be the same.

17. An action of G on a set S is transitive if the orbit of an element of S is all of S. Find all transitive actions of the dihedral group.

18. Classify all transitive actions of a group, up to the notion of two actions being the "same" from 16.

19. If G acts on S_1 and and S_2, then it acts on the ordered pairs $S_1 \times S_2$ by doing the original actions in each coordinate. Even if the actions on S_1 and S_2 are transitive, the action on $S_1 \times S_2$ might not be. For your transitive actions of the dihedral group in the previous question, determine how the products of the transitive actions decompose in terms of transitive actions.

20. Show that if a prime number p divides the order of a group G, then G has a subgroup of order p.