

Complex Numbers 2

Advanced Math Circle

February 26, 2017

Last week we had an introduction to complex numbers, and specifically we showed that complex numbers are a lot like vectors in a lot of ways. Complex number arithmetic was exactly the same as vector arithmetic with one big difference. Complex number multiplication was not the same as vectorial multiplication. Let's see if we can find a connection between complex number multiplication and vector geometry.

1. For all of the following, rotate means rotate counterclockwise. Plot each point as you compute it.
 - (a) Compute the result of rotating the vector $(5, -3)$ π radians around the origin.

- (b) What do you get from rotating the vector $(1, -\sqrt{3})$ by $\frac{\pi}{2}$ radians.

(c) Compute the result from rotating the vector $(0, 1)$ by $\frac{\pi}{2}$ degrees 3 times.

(d) What do you get if you multiply $(-1, -\sqrt{3})$ by 4 and rotate it by $\pi/3$ radians?

2. Now, let's multiply some complex numbers. For each multiplication, plot the two complex numbers, and the result of the multiplication.

(a) Compute $(5 - 3i)(-1 + 0i)$.

(b) Compute $(1 - \sqrt{3}i)(0 + 1i)$.

(c) Compute $(0 + 1i)^4$.

(d) Compute $(-1 - \sqrt{3}i)(2 + 2\sqrt{3}i)$.

3. Before we find a connection between complex number multiplication and vector operations, let's talk about something called the polar representation of a point in \mathbb{R}^2 .

What are all of the ways that you know to describe a point on the plane? Until now, chances are good that the only tool that you have is putting a Cartesian grid on the plane, and describing a point by talking about its x and y coordinate. That's a fine way to describe a point, but it isn't the only way.

Another way to describe a point on the plane, is to say how far it is away from the origin, and in what direction. This may sound strange at first, but it's the way that most people communicate directions to each other in real life.

For each of the following questions, compute two things, the distance that each point is from the origin, and the angle that each point makes from the x -axis. These two number can be put in a pair (r, θ) where $r \geq 0$, and $0 \leq \theta < 2\pi$, like $(1, \frac{\pi}{2}), (5, \frac{3\pi}{4})$.

- (a) Compute the polar form of $5 - 3i$, $-1 + 0i$ and $(5 - 3i)(-1 + 0i)$.

- (b) Compute the polar form of $1 - \sqrt{3}i$, $0 + 1i$ and $(5 - 3i)(-1 + 0i)$.

(c) Compute the polar form of $0 + 1i$, $(0 + 1i)^2$, $(0 + 1i)^3$ and $(0 + 1i)^4$.

(d) Compute the polar form of $-1 - \sqrt{3}i$, $2 + 2\sqrt{3}i$ and $(-1 - \sqrt{3}i)(2 + 2\sqrt{3}i)$.

(e) Look back at 1.a) – 1.d), 2.a) – 2.d) and 3.a) – 3.d). What do you notice?

(f) Can you think of a geometric interpretation of complex number multiplication?

(g) Is the Cartesian representation of a point unique? Is the polar representation unique? (Note, a proof isn't necessary, but write down some convincing arguments to support your claim)

(h) Based on your work on the previous problems, make an educated guess about what you get when you multiply a point (r_1, θ_1) by another point (r_2, θ_2) . What about adding two polar points?

4. Is there an easy way to convert from the polar representation of a point to the Cartesian one and vice versa? Yes and no. You can, but it's not exactly the most beautiful formula in all of mathematics. We won't use it, but in case you are curious, here it is. To convert from Cartesian to polar you have:

$$r = \sqrt{x^2 + y^2} \quad (1)$$

$$\theta = \arctan \frac{x}{y} \quad (2)$$

And to compute the Cartesian representation from the polar, use:

$$x = r \cos(\theta) \quad (3)$$

$$y = r \sin(\theta) \quad (4)$$

Solve the following problems.

- (a) Prove that multiplication by complex number z such that $|z| = 1$ is equivalent to a rotation about the origin.

- (b) Given a complex number z , can you figure out what \sqrt{z} is geometrically using the polar form?

(c) Do the same for the $\sqrt[3]{z}$.

(d) Using the polar form, reason why there should be exactly 3 unique solutions to the equation $z^3 = 1$.

(e) Extend the argument from the above question to prove that there should be exactly n unique solutions to the polynomial $z^n = 1$. The solutions are often called the 'roots of unity.'

(f) Start with a convex quadrilateral, and draw four squares, where each square shares one side with the quadrilateral. Let c_1, c_2, c_3 and c_4 represent the centers of those four squares in order. Prove that the lines c_1c_3 and c_2c_4 are of equal length, and are perpendicular. (Check with an instructor to make sure that you understand the statement of this problem).

(g) Prove that the sum of all n 'th roots of unity is zero. Interpret this geometrically.

5. One useful way to write the polar form is to use the representation $(r, \theta) = re^{i\theta}$. Proving that this representation is actually correct is WAY beyond the scope of the math circle, but you should just assume that it correct for now. One of the implications of this representation is the following formula:

$$re^{i\theta} = r \cos(\theta) + ri \sin(\theta) \tag{5}$$

(a) Prove that the $re^{i\theta}$ representation is at least plausible by proving that symbolic multiplication of two numbers in that representation is what you would expect it to be based on your work on problem 3.h)

(b) Use equation (5) to prove the (extremely) famous identity: $e^{i\pi} + 1 = 0$.

(c) Suppose we consider the $n - 1$ diagonals of a regular n -gon inscribed in a unit circle obtained by connecting one vertex with all the others. Show that the product of their lengths is n .

(d) What is wrong with the following argument?

$$\begin{aligned} -1 &= i^2 \\ &= \sqrt{-1}\sqrt{-1} \\ &= \sqrt{(-1)(-1)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

(e) The Chebyshev polynomials on the interval $-1 \leq x \leq 1$ can be defined as $T_n(x) = \cos n \arccos x$. Prove that the Chebyshev polynomials are indeed polynomials in x , even if they don't look like it at first glance. (Ask an instructor if you don't know what $\arccos \cdot$ is.)

- (f) Prove that if the consecutive vertices z_1, z_2, z_3, z_4 of a quadrilateral lie on a circle, then $|z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_1 - z_4|$.

- (g) Let $\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \dots$. By using some very, very clever arguments, show that ζ obeys the function equation:

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) (-z)! \zeta(1-z) \quad (6)$$

Where $!$ is an extension of the factorial to be defined on non-integer values.

- (h) Using the above problem, prove or disprove the claim that if $\zeta(z) = 0$, then z must be of the form $\frac{1}{2} + bi$ for real numbers b . (Hint, don't bother asking the instructors (or anyone else on earth for that matter) for help on this problem.)