Vectors

Advanced Math Circle

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Today we are going to spend some time studying vectors. Vectors are not only ubiquitous in math, and also used a lot in physics and engineering. Vectors are useful because we can do algebra with them just like regular numbers, but they describe things that are in more than one dimension.

The simplest way to think of a vector, is that it is a part of a line segment with a start and end point, and a direction. The notation for a vector is:

$$\bar{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1}$$

Where x_1 and x_2 are scalars (just regular numbers), such as $4, -17/6, \pi, \ldots$. Notice that $\bar{v}$ has a little bar above it. This it notation to remind us that $\bar{v}$ is a vector, and not scalar. When you are writing down the vector, don't forget the over bar!

We say that x_1 is the first component of the vector, and x_2 is the second component. The components can be positive, negative, zero, whatever.

Given any two points, we can construct a unique vector which connects the two point. For example, if I had two point $A = (1, 3)$ and $B = (-3, 6)$, then the vector that connects them is:

$$\overrightarrow{AB} = \begin{bmatrix} 1 - (-3) \\ 3 - 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \tag{2}$$

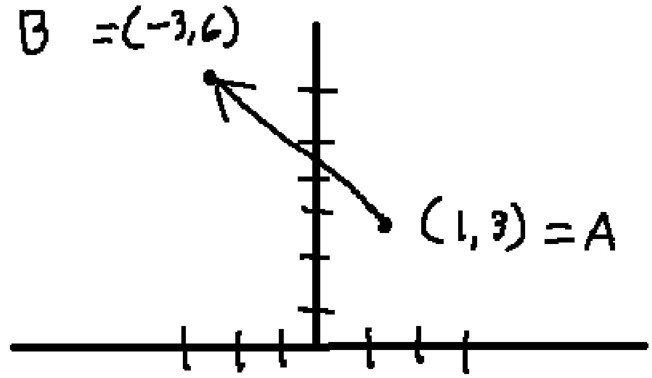
Notice that if we draw the points A and B on the Cartesian plane then the vector starting at A ends at B.

The length of a vector is $\|\bar{v}\| = \sqrt{x_1^2 + x_2^2}$.

1. Determine the following:

    (a) The vector starting at (1,1) and ending at (3,4).

Figure 1: The vector connecting the points A and B.



(b) The vector starting at (-3,-2) and ending at (3,4)

(c) The vector with the same direction at a) but twice as long.

(d) The vector with the same direction at a) but half as long.

(e) The vector with the same direction at a) but of length exactly 1.

2. Now, let's define the kind of algebra that we can do with vectors. If you have two vectors

$$\bar{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \bar{u} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{3}$$

then we define vector addition to be:

$$\bar{v} + \bar{u} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \tag{4}$$

And we can also define scalar-vector multiplication. If α (that's the Greek letter alpha) is a scalar, then:

$$\alpha\bar{v} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} \tag{5}$$

For the next few problems, let $\bar{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\bar{y} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\bar{z} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

(a) Calculate and then draw $\bar{x}, \bar{y}, \bar{x} + \bar{y}$

(b) Calculate and then draw $\bar{x}, \bar{z}, \bar{x} + \bar{z}$

(c) Calculate and then draw $\bar{x}, \bar{y}, \bar{x} - \bar{y}$

(d) Calculate and then draw $\bar{y}, 2\bar{y}$

(e) Calculate and then draw $\bar{x}, \bar{y}, \bar{z}, \bar{x} + 2\bar{y} - 3\bar{z}$

(f) Explain vector addition, subtraction and scalar-vector multiplication geometrically.

(g) Is it possible to change the direction of a vector by scalar multiplication? Explain.

(h) If I multiply a vector $\bar{x}$ by a scalar α, what is of $\|\alpha\bar{x}\|$ in terms of α and $\|\bar{x}\|$?

(i) Let the quadralateral Q have vertices at A, B, C, D. Let E, F, G, H be halfway in between AB, BC, CD and DA respectively. Prove that E, F, G, H forms the vertices of a parallelogram.

3. We can also define a kind of vector-vector multiplication, called the inner product. The inner product of two vectors gives you a SCALAR, NOT A VECTOR. It is defined as follows:

$$\langle \bar{v}, \bar{u} \rangle = x_1 y_1 + x_2 y_2 \tag{6}$$

Let $\bar{x}, \bar{y}, \bar{z}$ be the same as before.

(a) Compute $\langle \bar{x}, \bar{y} \rangle, \langle \bar{y}, \bar{z} \rangle, \langle \bar{z}, \bar{x} \rangle$.

(b) Is is always true that $\langle \bar{v}, \bar{u} \rangle = \langle \bar{u}, \bar{v} \rangle$ for all vectors $\bar{v}, \bar{u}$?

(c) Find an $\bar{x}, \bar{y}$ such that $\langle \bar{x}, \bar{y} \rangle = \|\bar{x}\| \, \|\bar{y}\|$

(d) Find another $\bar{x}, \bar{y}$ such that $\langle \bar{x}, \bar{y} \rangle = -\|\bar{x}\| \, \|\bar{y}\|$

(e) Find an $\bar{x}, \bar{y}$, neither of which are zero such that $\langle \bar{x}, \bar{y} \rangle = 0$

(f) Find at least three vectors $\bar{y}_1, \bar{y}_2, \bar{y}_3$ such that if $\bar{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then $\langle \bar{x}, \bar{y}_1 \rangle = \langle \bar{x}, \bar{y}_2 \rangle = \langle \bar{x}, \bar{y}_3 \rangle = 3$

(g) Explain how the previous problem means that it is impossible to define vector-vector division.

(h) Find a way to define the length of a vector, $\|\cdot\|$, in terms of the inner product.

4. One of the reasons that the inner product is useful, is that it gives you a very quick way of telling if vectors are parallel, or perpendicular. Specifically, $\bar{u}$ and $\bar{v}$ are parallel if and only if $\langle \bar{u}, \bar{v} \rangle = \|\bar{u}\| \|\bar{v}\|$. $\bar{u}$ and $\bar{v}$ are perpendicular if and only if $\langle \bar{u}, \bar{v} \rangle = 0$.

   (a) Prove the $\bar{u}$ and $\bar{v}$ are perpendicular if and only if $\|\bar{u} + \bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$. What is the more common name for this theorem?

(b) Define $proj_{\bar{u}}\bar{v} = \bar{u}\frac{\langle \bar{v}, \bar{u} \rangle}{\|u\|^2}$. Prove that $\bar{u}$ and $\bar{v}$ are parallel if and only if $proj_{\bar{u}}\bar{v} = \bar{v}$.

(c) Prove that $\bar{v}$ and $\bar{u}$ are perpendicular if and only if $proj_{\bar{u}}\bar{v} = \bar{0}$, the vector with all zero components.

(d) Prove that $\bar{v} - proj_{\bar{u}}\bar{v}$ is perpendicular to $\bar{u}$.

(e) Use the projection to define the area of a triangle which has two sides given by the vetors $\bar{v}$ and $\bar{u}$.

5. Finally, let's talk about liner dependence/independence. We say that a set of vectors $\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n$ are linearly dependent if there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ which are not all 0 such that:

$$\alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \cdots + \alpha_n \bar{v}_n = \bar{0}. \tag{7}$$

If a set of vectors is not linearly dependent, then it is linearly independent.

(a) Come up with sets of one, two and three vectors which are linearly dependent.

(b) Come up with sets of one and two vectors which are linearly independent.

(c) Are there any sets of three vectors in the plane which are linearly independent? Why?

(d) Prove that if a set of vectors is linearly dependent, then at least one of the vectors can be written as a linear combination of the other vectors in that set.

(e) We say that set of vectors $V = \{\bar{v}_1, \dots \bar{v}_m\}$ is a basis of $\mathbb{R}^n$ if the vectors in V are linearly independent, and if every vector in $\mathbb{R}^n$ can be written as a linear combination of vectors in V. Prove that every basis has the same number of elements. How is this related to the dimension of $\mathbb{R}^n$?