



(c) What about the reverse? If I give you a polynomial, how can you turn it into a strategy to beat this game?

(d) Here is a 'proof' that  $\pi$  is algebraic. Let's play the game. Start with  $\pi$ . Add 3, and square it. Then, take away 9, and then take away  $\pi(\pi + 6)$ . You should have exactly zero. This 'proof' shows that  $\pi$  is algebraic, but the video says that  $\pi$  was **not** algebraic. How can this be?! Resolve this paradox.

(e) One (amazing) fact is that there are lots of algebraic numbers which you can not write as a sequence of radicals, squares, etc... One such polynomial is  $x^5 - 4x + 2$ . Even though we can't write down any of its roots, show that it has at least 3 real roots.

2. Ok, now I want to talk about something which I think is amazing. Even though there are lots and lots of algebraic numbers, *almost all numbers are transcendental*. By that I mean that if you pick a point uniformly at random from an interval  $[a, b]$  where  $b > a$ , then your chances of picking an algebraic number are exactly zero, and your chances of picking a transcendental number is exactly 1. To explain more on this topic, we're going to watch a second video, called "Transcendental Darts" by the channel "Vihart".

<https://www.youtube.com/watch?v=Swm8tTLWirU>

- (a) As discussed in the video, one surprising fact is that the algebraic numbers are countable. In other words, there is a bijection between the algebraic numbers and the natural numbers. Use this fact to prove that there are an uncountable number of transcendental numbers.

- (b) The above result is kind of amazing, because it shows that compared to the number of numbers that there are total, the number of algebraic numbers is tiny even though almost all numbers that we interact with on a day-to-day basis are algebraic! The proof of this is not really difficult, but is tedious and not all that illuminating.

- (c) One of the things which is brought up in the Vihart video is that you could consider the set of all possible expressions that are any finite combination of symbols like  $2, 3, 0, +, -, \sqrt{\cdot}, \sqrt[3]{\cdot}, (\cdot)^3, \dots$  which are mathematically legal (so excluding things like  $(-)^5$  or  $\sqrt[4]{+--}$ ) then this set would be countable. Still, this set does not include every possible algebraic number. What are some algebraic numbers which would be missing from this set? \*Hint, this question is confusing, but not very hard. If the question statement doesn't make sense to you, then ask an instructor for help.

- (d) **Here is a proof sketch of the claim that the algebraic numbers are countable. First, recognize that every polynomial can be encoded by writing down its coefficients, therefore there is a bijection between all polynomials with integer coefficients, and all integer sequences of finite length. The set of all finite length integer sequences is countable (Proposition A). Further, each polynomial only has a finite number of roots, therefore the total number of root allowing for repetition is also countable (Proposition B). Prove propositions A and B.**

3. Now, I want to give you some properties of algebraic and transcendental numbers, and see if you can apply them to answer a pretty difficult question.

(a) Mathematicians say that a set  $\mathcal{A}$  is closed under some operation (like addition, subtraction, incrementation, exponentiation, etc...) if the following is true. If you start with elements in  $\mathcal{A}$  and only perform the operation specified, then you will always end up with other elements in  $\mathcal{A}$ . Can you show that the rational numbers are closed under addition, multiplication and integer exponentiation (i.e. taking something to an integer power), but are not closed under rational exponentiation (i.e. taking something to a rational power)? \*Hint, this question should be simple, once you understand what it means to be closed. As an instructor for help if this seems really hard.

(b) The algebraic numbers are closed under the normal algebraic operators like addition, exponentiation to a rational power, negation, etc... Showing that the algebraic numbers are closed under addition is actually pretty hard, but can you show that they are closed under negation, and exponentiation to a positive power?

(c) Now take for granted that the algebraic numbers are closed under addition, rational exponentiation, multiplication, negation etc... Which of the following *must* be true as well? Why or why not?

- The difference of two algebraic numbers is algebraic.

- The sum of an algebraic and transcendental number is transcendental.
  
- The difference of two transcendental numbers is transcendental.
  
- The multiplication of an algebraic and transcendental number is transcendental.
  
- The multiplication of two transcendental numbers is transcendental
  
- **If  $c$  is a transcendental number, then  $p(c)$  is also transcendental.**

(d) It is *extremely* hard to prove that a number is transcendental in general. Mathematicians have been able to come up with numbers that are transcendental, but it is very hard to prove that a transcendental number is such in general. A few notable examples of transcendental numbers are  $\pi$  and  $e$ , but no one knows if  $\pi + e$  or  $\pi e$  are transcendental or not. Use these facts, combined with the polynomial  $(x - \pi)(x - e) = x^2 - (\pi + e)x + \pi e$  to show that at least one of  $\pi + e$  or  $\pi e$  are irrational. \*Hint, what if they were both rational?

4. I want to conclude this lesson with some good old challenge problems.

- (a) You are given a very special (and unusual) hole puncher. This puncher punches out points on the x-y plane. If you hover this puncher over a point  $(x, y)$ , then it will punch out all points that are a transcendental distance away from the point  $(x, y)$ . How many punches do you have to make to punch out all points in the x-y plane?
- (b) What if the hole puncher punched out points which were an algebraic distance away from the point  $(x, y)$ ?
- (c) Let's play a game. I think of a polynomial with natural coefficients, which you try and guess. You give me a number, I plug that number into my polynomial, and tell you the result (in decimal). Prove that you can plug in 1 number, and figure out what my polynomial is in finite (but unbounded) time

(d) **Prove that if I plug in two numbers, then I can figure out your polynomial, and I can do it in finite time which is bounded by the answer to my first guess.**

(e) **Prove that if you pick a number uniformly at random from some interval of the form  $[-c, c]$  for  $c > 0$ , then the chances of picking an algebraic number is exactly 0.**