Math Circle
Assignment #1 – Walk the Dog

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Due

Problem 1. Suppose your local park is convex and contains a convex lake in the middle. It is clear that the area of the park is greater than the area of the lake, but what about the perimeter? Suppose you walk clockwise with your dog around the perimeter of the park and your friend walks clockwise with her dog around the perimeter of the lake, both with leashes of length x. What can you say about the area of the region your dog's path cuts out versus the area of the region your friend's dog cuts out? Use the formula,

$$A_x = A_0 + P_0 x + \pi x^2,$$

to show that the perimeter of the park is longer than the perimeter of the lake!

Problem 2. In America if you send a package in a box they charge you by the length of the longest side of the box. If I wanted to send a 1.5 meter pool cue you would think I must pay for a 1.5 meter box, but I can put the pool cue diagonally in a $1 \times 1 \times 1$ meter box and cheat the shipping system!

In the Russian shipping system they take the three edge lengths, a, b, c and measure your box by $a+b+c$. Can you cheat the Russian system by fitting a 'larger' box inside a 'smaller' box? (Hint: if $X \subseteq X'$ then $V_x \leq V'_x$ for every x.)

Problem 3. More generally, show that if we have two convex solids, X, X' and $X \subseteq X'$ then we must have $\ell_0 \leq \ell'_0$, where ℓ_0 is the linear measure discussed in the talk. Thus if the Russian shipping system wanted to measure an object they would determine ℓ_0 of the object and charge based on that.

Problem 4. Consider a flat disk, D of radius a in space. If we take D_x to be the set of points which have distance less than x away from D then what is D_x geometrically?

Problem 5. In the previous problem you saw the shape D_x was a 'donut with the hole filled in'. If we take the set of points distance x away from the *perimeter* of the disk then we get an actual donut! Actually, we only get a donut if x is small enough. How small must x be so that the donut we just described still has a hole in it?

Problem 6. Calculate the volume of the donut from the previous problem as follows,

- Take a cylinder of length $2\pi a$ and radius x. Halfway along the length of the cylinder make a $45°$ cut and twist the top half so you have an L-shaped 'pipe'.
- Do the same thing at one quarter and three quarters of the length of the cylinder to get a square shaped pipe.
- Now imagine making n cuts equally along the length of the cylinder each with angle $\frac{360°}{2n}$. Now twist each piece so the pipe makes a $\frac{360°}{n}$ bend at every cut. What does your shape look like?
- As n increases the pipes that we make get closer and closer to the shape of the donut while still having the volume of the original cylinder. What is the volume of the donut?

Problem 7. Calculate the volume of D_x from the above problems by using the formula,

$$V_x = V_0 + S_0 x + \ell_0 \pi x^2 + (4/3)\pi x^3$$

Calculate ℓ_0 by approximating the disk by polyhedra P_n and finding ℓ_0 for each of these polyhedra.

Problem 8. What is the volume of the part of the donut that lies inside the cylinder which extends above and below the disk? What about the part of the donut that lies outside the cylinder? Is this *surprising?* Show more generally that if a worm of radius r is eating his way through an apple then the amount of apple he has had to eat depends only on the distance he has travelled, and not on his path!

Problem 9. Let C be a convex shape in the plane with perimeter P_0. Define X to be the set of points in space which lie above C but no more than 1 unit away from C, *i.e.* if C is a circle then X is a cylinder, If C is a square then X is a cube. Calculate ℓ_0 for the solid X.

Problem 10. Can you calculate ℓ_0 for a parallelelpiped? Given a polyhedra we can imagine that its edges are toothpicks and its vertices are marshmallows. If we treat the cube like this then you see we can deform the cube by smushing it flat without breaking any toothpicks or detaching them from the marshmallows. Show that ℓ_0 is unchanged under these kinds of deformations.

For the next few problems we need the following definition. Call two polygons *scissors equivalent* if you can cut one into finitely many pieces using straight cuts and rearrange those pieces into the other polygon. For instance any polygon you can make with tangrams are scissors equivalent to the square they are cut out of.

Problem 11. Show that a right triangle is scissors equivalent to a rectangle with the same base length. Now show that any triangle is scissors equivalent to some rectangle.

Problem 12. Show that a rectangle is scissors equivalent to the square with that same area.

Problem 13. Show that we can always 'triangulate' a polygon, that is, cut it into finitely many triangles. (Hint: let m be a slope which does not occur as the slope of any edge of the polygon. Now cut along a line of slope m going through each vertex. What does your figure look like now?) Can you find another proof where you introduce no new vertices?

Problem 14. Show that two polygons are scissors equivalent if and only if they have the same area!