Combinatorics on the Chessboard

Interactive game:

1. On regular chessboard a rook is placed on $a1$ (bottom-left corner). Players A and B take alternating turns by moving the rook upwards or to the right by any distance (no left or down movements allowed). Player A makes the first move, and the winner is whoever first reaches $h8$ (top-right corner). Is there a winning strategy for any of the players?

Solution:
Player B has a winning strategy by keeping the rook on the diagonal.

Knight problems based on invariance principle:

A knight on a chessboard has a property that it moves by alternating through black and white squares: if it is on a white square, then after 1 move it will land on a black square, and vice versa. Sometimes this is called the chameleon property of the knight. This is related to invariance principle, and can be used in problems, such as:

2. A knight starts randomly moving from $a1$, and after n moves returns to $a1$. Prove that n is even.

Solution: Note that $a1$ is a black square. Based on the chameleon property the knight will be on a white square after odd number of moves, and on a black square after even number of moves. Therefore, it can return to $a1$ only after even number of moves.

3. Is it possible to move a knight from $a1$ to $h8$ by visiting each square on the chessboard exactly once?

Solution:
Since there are 64 squares on the board, a knight would need 63 moves to get from $a1$ to $h8$ by visiting each square exactly once. However, based on the chameleon property, after 63 moves it has to be on a white square, while $h8$ is a black square, therefore, such path is impossible.

Problems related to placing pieces on the chessboard:

4. Find the maximum number of specific chess pieces you can place on a chessboard such that none of them is under attack. Solve it for:
a) rooks
b) queens
c) bishops
d) kings

e) knights

In this class we will discuss solutions for a and e. Whoever is interested, try solving b,c and d using similar approach. We can go over those later if there is any interest.

For these problems first we need to find a cap, then need to show an example of a configuration using that number.

In many board problems an important technique is to consider the problem for a smaller board first. If we can divide the board into smaller regions such that there is a clear cap for each subregion, then by adding those numbers for each subregion we can get a total cap.

Solution for a):
Consider each vertical line of the board ($a1$ to $a8$, $b1$ to $b8$, etc). There are 8 lines, and clearly you can place at most 1 rook on each line. Therefore, you can place at most 8 rooks on the chessboard. We still need to show that it is possible to place 8 rooks on a chessboard without attacking each other. Indeed, one such example is placing 8 rooks on a main diagonal. Therefore, the maximum number is 8.

Solution for e):
Divide the board into eight 4×2 regions. Each region can have at most 4 knights, therefore, full chessboard can have at most 32 knights. And as an example, place 32 knights on all 32 black squares (or all 32 white squares), then none of them will attack each other. So the answer is 32. It turns out that placing knights on same color squares are the only two ways that you can place 32 knights without attacking each other.

Below is another take home problem (harder version of $4a$):
5. What's the maximum number of rooks you can place on a chessboard such that each rook is under attack by no more than one other rook?

6. Find a minimum number of specific chess pieces needed to place on a chessboard in a way that all free squares of the board are under attack. Solve it for:
a) rooks
b) queens
c) bishops
d) kings
e) knights

Note that in those problems the requirement is to keep under attack only empty squares; if there is a piece on it, then it doesn't have to be attacked. There are versions of those problems with that extra requirement as well.

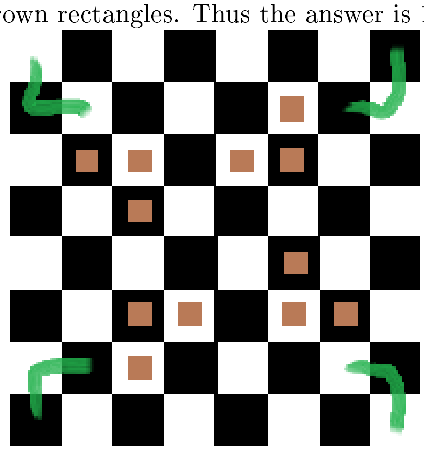
In this class we will discuss b and e (a, c and d to think about at home).

Solution for b):

Consider 5 queens placed at $a2, c4, d5, e6$ and $g8$. Then note that every square of the board will be under attack. Therefore, 5 queens are sufficient for the problem requirement. In order to show 5 is the minimum number, one needs to show that 4 queens cannot cover the whole board. It turns out that's actually very hard to show by hand; I wasn't able to find any proof for this. Instead, it is done by writing computer code that looks at various permutations of 4 queens on a chessboard and in each case finds a square that is not under attack.

Solution for e):

Consider the 12 squares with a green line: $a1, a2, b2, a7, a8, b7, g2, h1, h2, g7, h7, h8$. Note that a single knight cannot attack or cover more than 1 of those squares, thus you need at least 12 knights to attack or cover those 12 squares. And with 12 knights you can actually attack the whole board (empty squares), illustrated with brown rectangles. Thus the answer is 12.



Now let's look at some counting problems related to rooks.

7.

a) What's the number of ways to place 8 rooks on a chessboard such that no rook is under attack?

b) What's the number of ways to place 8 rooks on a chessboard such that no rook is under attack and there are no rooks placed on the main diagonal ($a1 : h8$)?

c) What's the number of ways to place 8 rooks on a chessboard such that each square of the board is under attack?

Solution for a):

Note that each row contains one rook. The first rook (on the first row) can be placed on any of the 8 columns; second rook can be placed on 7 columns, etc, and last rook can be on a single column. The total number of ways is, therefore, 8!. In combinatorics terms, this is equivalent to number of permutations of a set with 8 elements.

Solution for b):

In this case we are dealing with number of derangements of a set, i.e. permutations where no element appears in its original position. The number of derangements satisfies the following recursive equation:

$D(n) = (n-1)(D(n-2) + D(n-1))$

In case of rooks and a chessboard ($n = 8$), this can be explained as follows: consider the first rook; it can be placed on 7 different columns. Assume it is on column k. Then consider the k-th rook. If it is placed on the first row, then we're left 6 rooks, which gives us $D(6)$ derangements. If k-th rook is not on the first column, then we're left with $D(7)$ derangements. Thus:

$D(8) = 7 * (D(6) + D(7))$

By induction it can be easily proved that $D(n)$ also satisfies equation:

$D(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$

Using that formula we can get $D(8) = 14833$.

Solution for c):

Note that in order for each square of the board to be under attack, there must be a rook on either every row or every column. Indeed, if there was a row without a rook, and a column without a rook, then their intersection would not be under attack. The number of configurations where there is 1 rook on each row is 8^8 (because each one of the 8 rooks can be placed in 8 different places). Similarly, the number of configurations covering each column is also 8^8. And the number of configurations where there is a rook on each row and each column is the same as $7a$, i.e. 8!. Thus, the answer is $2 * 8^8 - 8!$.

Below is another take home problem:

8. Is it possible to cover all of the chessboard squares with knight moves by visiting each square exactly once?

Second part of the class is related to chessboards in general, without any chess pieces involved.

Many board problems are related to tiling and coloring. Domino tiles are pretty common:

9. Corners $a1$ and $h8$ have been removed from a chessboard. Can you cover the rest with 1×2 dominoes?

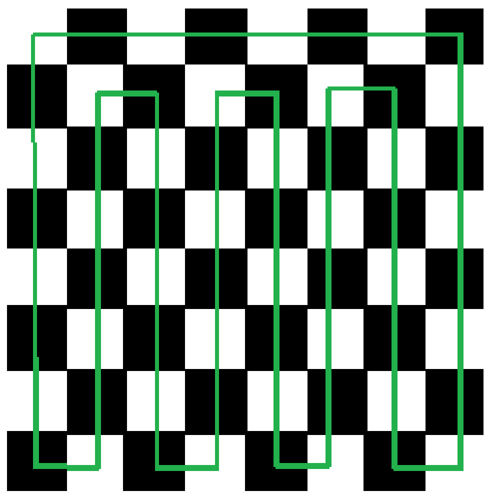
Solution:

Here we use an approach similar to problems related to knight movements. Here an important property of a domino is that it covers 1 white square and 1 black square. This is also a version of invariance principle (the difference between white and black cells remains constant). Note that by removing $a1$ and $h8$, we're left with 30 black squares and 32 white squares. Therefore, the board cannot be covered with dominoes, because dominoes can cover only equal amount of black and white squares.

10. Two arbitrary squares of different colors (i.e. 1 black square and 1 white square) have been removed from a chessboard. Prove that the rest of the board can always be covered with 1×2 dominoes.

Solution:

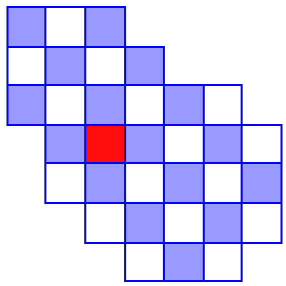
Below is an elegant proof by Ralph E. Gomory. Consider the following closed path on a chessboard (the green lines):



If two squares of different colors have been removed, then we can travel from the first removed square to the second one in two directions, and each path will contain even number of squares, thus they can be covered with dominoes, and as a result the full board will be covered.

Now let's consider arbitrarily shaped boards (not necessarily rectangular). It is interesting to see which boards can be covered with dominoes, and which cannot. As we saw, if a board contains unequal number of black and white squares, then it cannot be tiled with dominoes. So it is more interesting to consider boards with equal number of black and white squares.

11. Can the following board be tiled with dominoes (the red square has been removed)[1]?



Solution:

Although there is an equal number of black and white squares on the board, the answer is still no. Consider the first two black squares of top three rows (6 squares total). We need at least 6 dominoes to cover these. However, note that combined they have only 5 different neighbor squares. Thus, those 6 dominoes would have to overlap, meaning the tilling is impossible.

In general, if on a board we can find k squares of one color with less than k total neighbors, then the board cannot be covered with dominoes. The proof follows from the Pigeonhole principle.

It turns out the opposite is also true: any board or region that cannot be tiled with dominoes contains k cells of one color with fewer than k neighbors (proof by Philip Hall; this is a special case of marriage theorem). Thus, in order to show that a domino tiling is impossible for a given board, we can try looking for such cells.

Take-home problem:

12. Corner $a1$ has been removed from the chessboard. Can you tile the rest with 1×3 "trominoes"?

Now let's consider tiling boards of rectangular shape with other rectangles.

13. Is it possible to tile[2]:

a) 7×10 board with 2×3 rectangles

b) 17×28 board with 4×7 rectangles

c) 10×15 board with 1×6 rectangles

Solution:

a) There are 70 squares on the board, while each rectangle covers 6 squares, so clearly the covering is impossible.

b) Note that the previous argument doesn't apply here. We use a different approach: consider the first row of board; it contains 17 squares. Each 4×7 rectangle can cover either 4 or 7 of those. However, 17 cannot be written as a sum of fours and sevens. Therefore, even the first row cannot be covered, which makes the board covering impossible.

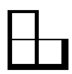
c) Here the previous two arguments don't work anymore. It turns out that tiling is still impossible, because neither 10 nor 15 is divisible by 6. This follows from a theorem by de Bruijn and Klarner (and so do parts a and b):

An $m \times n$ board can be tiled with $a \times b$ rectangles if and only if:

- mn is divisible by ab
- both m and n can be written as $ax + by$ where x and y are non-negative integers
- either m or n is divisible by a, and either m or n is divisible by b

Below is another problem related to chessboard tiling:

14. A random square has been removed from a chessboard. Prove that you can cover the rest with angled trominoes:



Solution:

As noted in some of the chess problems (e.g. $4e$), it is often useful to solve a board problem for smaller regions first. In fact, this problem can be solved using mathematical induction for any board with dimensions $2^n \times 2^n$ (for regular chessboard, $n = 3$).

- if $n = 1$, then we have a 2×2 board, and the solution is trivial

- assume the statement is true for $n = k$, i.e. it is possible to cover a $2^k \times 2^k$ board by removing any square. Consider a $2^{k+1} \times 2^{k+1}$ board. By cutting it through the middle vertically and horizontally, we get four $2^k \times 2^k$ boards. When we remove a square from the original board, it has to be in one of those smaller boards. Based on the induction hypothesis, that board can be covered with angled trominoes. For the remaining 3 subregions, note that you can place a tromino on the center of the original board such that it covers 1 corner from each of those 3 subregions. That gives us another 3 boards with 1 square missing. Those can also be covered based on the induction hypothesis, hence the original board can be covered too.

Below are several chessboard related problems from various AMC stages, leading to IMO[3]. Note that there is a jump in difficulty level; USAMO and IMO are generally harder than earlier stages of AMC, and there is more time allocated for each problem.

Easy:

15. From 2009 AMC 8:

On a regular chess board, what is the probability that a randomly chosen unit square does not touch the outer edge of the board?

16. From 2012 AMC 10A:

How many black squares are on a 31×31 chessboard?

Hard:

17. From 1998 USAMO:

Consider a 98×98 chessboard on a computer screen. One can select with a mouse any rectangle with sides on the lines of the chessboard, and switch the colors in the selected rectangle (black becomes white, white becomes white). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

18. From 1976 USAMO:

a) Suppose that each square of a 4×7 chessboard is colored either black or white. Prove that with any such coloring, the board must contain a rectangle whose four distinct unit corner squares are all of the same color.

b) Show that the above statement is false for a 4×6 chessboard, i.e. it can be colored in a way that the four corner squares of every rectangle are not all of the same color.

19. From 2014 IMO:

Consider an $n \times n$ chessboard, where $n \geq 2$. A configuration of n rooks is called *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook.

If there is any interest, we can go through the solutions of those problems in other classes.

References

[1], [2]: Federico Ardila, Richard P. Stanley: *Tilings*
[3] artofproblemsolving.com

Literature

Evgeny Gik. *Math and Chess*