# Fractions and Decimals

Los Angeles Math Circle (Beginner)

October 14, 2018

1. **Warm-up problems** To warm up for today, let's practice some exponents and powers! Calculate the following:

   (a) $2^2$

   (b) $2^2 \times 2^3$

   (c) $\frac{2^4}{2^2}$

   (d) $\frac{2^{2^2}}{4^4}$

   (e) $(x \cdot y)^3$

   (f) $(x^2 \cdot y)^2$

2. Simplify the following

   (a) $\frac{3^{20}7^6}{3^2 7^2 5^2 2^2}$

(b) $\frac{u^4 x^3}{(u^2 \cdot v)^4}$

3. What is larger, $2^{3^4}$ or $4^{3^2}$? Why?

4. Find the value of $x$ in $\frac{4}{5} = \frac{32}{x}$. Hint, use cross multiplication.

5. Alright, enough warm-up. Let's move on to the fraction and decimal problems. Please find the least common multiples for each of the following numbers.

   (a) 2 and 3.

   (b) 4 and 16.

(c) 11 and 12.

(d) $n$ and $n+1$.

(e) 5, 7 and 11.

(f) $n$, $n^2$ and $n^3$.

6. One of your friends **really** likes fractions and has offered you a series of trades. For each of the following, if the trades are in your favor or not, and do it without converting to decimals. All of the fruits are identical.

   (a) She'll give you $\frac{3}{10}$ of her apple for $\frac{5}{14}$ of yours.

(b) She'll give you $\frac{9}{12}$ of her watermelon if you give her either $\frac{11}{15}$ or $\frac{15}{20}$ of yours.

(c) She'll give you $\frac{43}{44}$ of her mango if you will give her $\frac{43}{3}$ of $\frac{1}{15}$ of one of yours.

7. Please find two fractions that are in-between $\frac{2}{3}$ and $\frac{4}{5}$.

8. Convert the following fractions into decimals and say which one is larger.

   (a) $\frac{7}{4}$, $\frac{42}{25}$.

(b) $\frac{3333}{500}$, $\frac{667}{100}$.

(c) $\frac{9}{4}$, $\frac{226}{1000}$.

9. Please convert the following fractions into decimals

(a) $\frac{1}{3}$.

(b) $\frac{2}{13}$. *Hint, $\frac{1}{13} = 0.\overline{076923}$

(c) $\frac{6}{7}$. *Hint, $\frac{1}{7} = 0.\overline{142857}$.

(d) $\frac{1}{9}$.

(e) $\frac{12}{99}$.

(f) $\frac{123}{999}$.

(g) $\frac{12345678}{99999999}$.

10. How do your answers for 8 and 9 differ? What do all of the answers to 8 have in common that is different from your answers to 9?

11. Find the prime factors of all of the following numbers.

    (a) 4

    (b) 25

    (c) 100

    (d) 500

    (e) 1000

    (f) 3

(g) 7

(h) 13

(i) 9

(j) 99

(k) 999

12. Notice that the numbers that you factored in question 11 were the denominators of the fractions in problem 8 and 9. Make a two column table. In the left column, write down the prime factors of the denominators of problem 8. In the right column do the same for problem 9. What do you notice? Check with an instructor to make sure that you have it correct.

13. Come up with your own name for this property. Try and be creative!

14. Without converting to a decimal, circle all of the numbers that have terminating decimal expansions (i.e. don't go on forever).

$$\frac{1}{7} \quad \frac{3}{5} \quad \frac{64}{125} \quad \frac{13}{30} \quad \frac{7}{35} \quad \frac{19}{256} \quad \frac{11}{160} \quad \frac{57}{5} \quad \frac{1}{300} \quad \frac{37}{1200} \tag{1}$$

15. Can you think of a fraction whose decimal expansion neither repeats itself forever, nor terminates after finite digits?

16. Can you think of a decimal number that neither repeats itself nor goes on forever?

17. Do you think that the following decimal number can be written as a fraction? Make an educated guess, you don't have to prove it.

$$0.11010010001000010000001\ldots \tag{2}$$

18. Convert the following decimals into fractions. What do you already know about the fractions without having to compute anything?

    (a) 0.08

    (b) 0.105

    (c) 0.25

(d) 0.125

19. So now we have a pretty good idea about what makes a fraction terminate or not, but we don't know for sure, we just have a good idea. To be sure, let's try and *prove* it!

    (a) Prove that if a fraction has a denominator of $2^n 5^n$ it can always be written as a decimal that terminates, no matter what whole number $n$ is. Ask an instructor to check your answer once you have an idea.

    (b) Prove that if a fraction has a denominator of $2^n 5^{n+k}$ it can always be written as a decimal that terminates, no matter what whole number $n$ and $k$ are. Have an instructor check your work. *Hint, change this fraction into something that looks like your answer to part (a).

(c) Prove that if a fraction has a denominator of $2^{n+k}5^n$ it can always be written as a decimal that terminates, no matter what whole number $n$ and $k$ are. Have an instructor check your work. *Hint, change this fraction into something that looks like your answer to part (a).

(d) What did we prove? Explain it in your own words.

20. Now let's try and prove that no other fractions have terminating decimal expansions other than the ones that we just discussed.

(a) Think of your most favorite example of a fraction less than one whose denominator has factors other than 2 and 5.

(b) Let's pretend for a second that the fraction did have a finite decimal expansion. Then it would look something like

$$\frac{a}{b} = 0.c_1c_2c_3\ldots c_n \tag{3}$$

for whole numbers $a, b$ and $n$, and digits $c_1, \ldots, c_n$. What happens if we multiply both sides of the equation by $10^n$?

(c) I claim that the equality you just wrote can not be true unless either $b$ divides $a$, or else $b$ divides 10. Why am I correct?

(d) What did we just show? Explain in your own words.