Casting out Nines, and other kinds of Math Magic

Junior Level Math Circle

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1 An introduction

Let's start by looking at a simple magic trick.

Problem 1. Pick a number between 1 and 10. Then perform the following steps (show your work on the side!)

(i) Take your number, and multiply it by 2

(ii) Take that number, and subtract 1 from it

(iii) Take that number, and multiply it by 9

(iv) Then take the digits of the number, and add them together. For instance, if you 34, you would now have $3 + 4 = 7$

(v) Look at the phrase below. Pick the letter in the phrase corresponding to the number you now have.

Like a pony he ran away

For instance, if you had 7 at this point, you would pick the letter *o*.

(vi) Think of a color that begins with that letter.

When both you and your neighbors have gotten their colors, compare you answer with your neighbor. Did you have the same color? How did the magic trick work?

In this class, we will look at how this magic trick works.

2 Relations

What does that mean, the word "related". It means we look at a pair of things, and decide that they are somehow special together. Let's look at a few examples of relations.

- $a = b$ is a relation that we can have between two numbers
- $a < b$ is a relation between two numbers.
- *" Is sitting on the left of"* is a relation that we can have between students.
- *"matches with"* is a relation that we can have between clothes.

A relation is a comparison that we can have between two different objects. Here are some properties that we can give to a relation

(i) **Reflexivity** $a \sim a$, (a is related to a)

(ii) **Symmetry** If $a \sim b$ then $b \sim a$ (If a is related to b, then b is related to a)

(iii) **Transitivity** If $a \sim b$, and $b \sim c$, then $a \sim c$.

Problem 2. Look at the example relations and identify which ones have the following properties.

(a) Which of the relations are Reflexive?

(b) Which of the relations are Symmetric?

(c) Which of the relations are Transitive?

Problem 3. Can you construct a relation that is transitive, but not symmetric or reflexive?

If a relation satisfies all three of the properties, we call it an equivalence relation.

When we have an equivalence relation, we can talk about groups of elements together. An **equivalence class** is a set of elements that are all related. Let us look at an example.

Example 1. *Let us begin with the set of animals, and we say that two animals are related if they have the same number of legs. For example, dogs and cats are related, but cats and fish are not. This forms an equivalence on the set of animals. The set of all animals with two legs form one equivalence class, and the set of all animals with 4 legs forms a different equivalence class. We belong to the class of animals with two legs.*

3 Modular Arithmetic

We are going to define a new equivalence relation on numbers. We say that two numbers, a and b are **congruent mod** n if their difference, $a - b$ is divisible by n. If two numbers are congruent mod n, then we will write

$$a \equiv b \mod n$$

Let us look at some examples.

- $1 \equiv 3 \mod 2$. In fact, all odd numbers are congruent $1 \mod 2$.
- $3 \equiv 13 \mod 10$.
- If a divides b evenly, then $b \equiv 0 \mod a$. Why?

Now you check some!

Problem 4. For each number a, find the number between 0 and 4 that it is equivalent to.

(a) $29 \equiv \quad \mod 5$

(b) $16 \equiv \quad \mod 5$

(c) $19 \equiv \quad \mod 5$

(d) $501 \equiv \quad \mod 5$

Problem 5 (Working $\mod 2$)**.** For each number a, find the number between 0 or 1 that it is congruent to.

(a) What is $3 \equiv \quad \mod 2$.

(b) What about $5 \equiv \quad \mod 2$.

(c) Why are all odd number congruent to $1 \mod 2$?

Problem 6 (Working $\mod 10$)**.** For each number a, find the number between 0 and 9 that it is congruent to.

(a) What is $38 \equiv \quad \mod 10$?

(b) What about $22 \equiv \quad \mod 10$?

(c) What is an easy way of telling what something is congruent to $\mod 10$?

Let us check that $\mod n$ is actually an equivalence relation. We must show that congruency satisfies the three properties: Reflectivity, Symmetry and Transitivity.

(i) **Reflectivity** $a \equiv a \mod n$. This is because $a - a = 0$ and of course, n divides 0.

(ii) **Symmetry** If $a \equiv b \mod n$, then n divides $-(a - b)$, which is just $b - a$. Therefore, $b \equiv a \mod n$

(iii) **Transitivity** If $a \equiv b \mod n$ and $b \equiv c \mod n$, we have that n divides $a - b$ and n divides $b - c$. Therefore, n divides

$$(a - b) + (b - c)$$

We have that

$$(a - b) + (b - c) = a - c$$

As n divides $a - c$, the congruence $a \equiv c \mod n$ holds.

When we say two numbers are congruent $\mod n$, we mean that they belong to the same equivalence class $\mod n$. We now ask: if $a \equiv p$, and $b \equiv q$, what is $a + b$ congruent to? I claim that it is congruent to $p + q$. What about ab?

Theorem 1. *If* $a \equiv p \mod n$ *and* $b \equiv q \mod n$*, then* $a + b \equiv p + q \mod n$*.*

Proof. If $a \equiv p \mod n$, then

$$a - p$$

is divisible by n. Since $b \equiv q \mod n$, then

$$b - q$$

is divisible by n. We therefore have that n divides

$$a - p + b - q$$

By rearranging the terms in the equation, we have that

$$(a + b) - (p + q)$$

is divisible by n. Therefore (by the definition of congruence) we have that $a + b \equiv p + q$.

□

Theorem 2. *If* $a \equiv p \mod n$ *and* $b \equiv q \mod n$*, then* $ab \equiv pq \mod n$*.*

Proof. We know that

$$a - p = nk$$

because $a - p$ is divisible by n. We also have that

$$a - q = nl$$

Therefore, $a = nk + p$ and $b = nl + q$. It follows that

$$\begin{aligned} ab - pq &= (nk + p)(nl + q) - pq \\ &= nknl + nkq + pnl + pq - pq \\ &= n(knl - kq + pl) \end{aligned}$$

which shows that n divides $ab - pq$, which shows that $ab \equiv pq \mod n$. □

In short: the sum of mods is the mod of the sum, and the product of the mods is the mod of the product.

Problem 7 (Use mod 7)**.** Isaac is going to have a barbecue next weekend. He buys hot dogs and he has hotdog buns at home. Hot dogs come in packages of 7. Isaac has a rule at his house, and the rule is you cannot eat a hot dog without a bun (or vice versa)

(a) Suppose that Isaac has 34 hot dog buns. How many packages of hot dogs should he buy if he doesn't want to have left over hot dogs? How many hot dog buns will he be left over with?

(b) The very same week, Jonathan decides to have a Bratwurst party. Coincidentally, Bratwurst also is packaged in groups of 7. He has the same rule about eating bratwurst and using buns. If he starts with 22 buns, how many packages of bratwurst should he buy? How many buns will he be left over with?

(c) That weekend, Jonathan and Isaac have discovered that they are holding the barbecue in the same the same park. Why is this a good thing?

Problem 8 (Use mod 4)**.** Jeff is doing laundry, and needs quarters to pay for the laundry machine. Laundry costs $1.25 a load. He has to go to the arcade at UCLA to turn his dollar bills into quarters. As you know, the vending machine at the arcade takes dollar bills and turns them into 4 quarters.

(a) If Jeff has to do just one load of laundry, what is the smallest number of quarters that he will be left over with if he uses the arcade machine to make change?

(b) If Jeff has to do two loads of laundry, what is the smallest number of quarters he will be left over?

(c) Jeff goes on a vacation, and when he comes back, he finds he has 57 loads of laundry to do. How much change will he have left over when he does laundry?

4 The Trick Explained

The magic trick revealed. How did the trick work? The trick lies in the step where you add the digits together. When we write a number between 0 and 99, we can write it as a a sum of it's tens place and ones place. For example,

$$\begin{aligned} 25 &= 20 + 5 \\ &= 2 \times 10 + 5 \end{aligned}$$

What happens when we look at a number $\mod 9$?

$$\begin{aligned} 25 &\equiv 2 \times 10 \mod 9 \\ &\equiv 2 \times 1 + 5 \mod 9 \\ &\equiv 2 + 5 \mod 9 \end{aligned}$$

The sum of the digits of a number is congruent to the number $\mod 9$! Let us look closely at the magic trick. Let us say that we started with the number n, and keep track of what we get.

(i) Take your number, and multiply it by 2 $\quad n \mapsto 2n$

(ii) Take that number, and subtract 1 from it $\quad 2n \mapsto 2n - 1$

(iii) Take that number, and multiply it by 9 $\quad 2n - 1 \mapsto 9(2n - 1)$

(iv) Then take the digits of the number, and add them together.

This is the important step. As $9(2n - 1)$ is divisible by 9, we know that it is congruent to $0 \mod 9$. We also know that the sum of digits is also congruent to $9(2n - 1) \mod 9$. So we have that

$$9(2n - 1) \equiv 0 \equiv \text{sum of the digits of } 9(2n - 1) \mod 9$$

(v) Look at the phrase below. Pick the letter in the phrase corresponding to the number you now have. (which is either 9 or 18)

Like a pon**y** he ran awa**y**

Which letters have a position that is congruent to $0 \mod 9$? They are both y. So at this point, everybody has chosen the letter y.

Finally, the last step is to pick of a color that starts with the letter y. Anybody think of something other than yellow?

5 Further Problems

Problem 9. We know that if we multiply a number by 9 the sum of its digits is divisible by 9. Does the converse hold? That is, does a number's digits being divisible by 9 imply the number is also?

(a) $10 \equiv \quad \mod 9$

(b) $100 \equiv \quad \mod 9$

(c) $1000 \equiv \quad \mod 9$

(d) $4000 \equiv \quad \mod 9$

(e) $700 \equiv \quad \mod 9$

(f) $60 \equiv \quad \mod 9$

(g) $1 \equiv \quad \mod 9$

(h) $4761 \equiv \quad \mod 9$ [**Hint:** $4 \times 1000 + 7 \times 100 + 6 \times 10 + 1 = 4761$]

Problem 10. Does a similar trick work for 3? Why or why not?

Problem 11. Comparing 9 and 3

(a) $4444 \equiv \quad \mod 9$

(b) $9702 \equiv \quad \mod 9$

(c) $4444 \equiv \quad \mod 3$

(d) $3433 \equiv \quad \mod 3$

Problem 12. Counting Students

(a) Suppose all the students in Math Circle fit into some number of tables of seven and one table of five. How can this be written in modular form?

(b) One day math circle is in a different classroom where everyone is at tables of four except there are two tables of five. How can this be written in modular form?

(c) How many students are in the class if there are no more than 30 students?