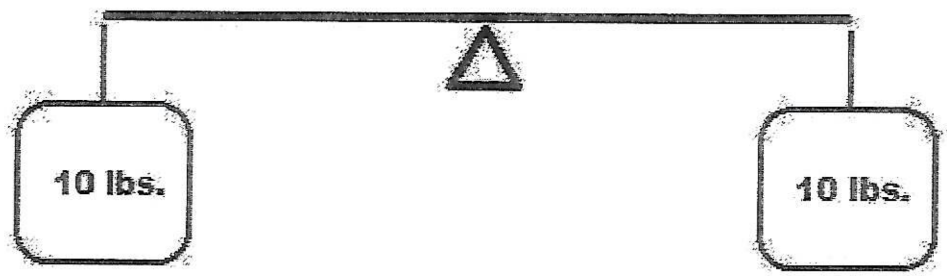
A LEVER PLAYING FIELD

MATH CIRCLE (BEGINNERS) 05/20/2012

You have a scale which has a rigid bar from which weights can be hung. The point at the center on which the scale balances is called the fulcrum. Use your intuition and experience to answer the following:



(1) If you have **equal** weights at **equal** distances from the fulcrum, the scale is:

(a) tilted down on the left

(b) balanced

(c) tilted down on the right

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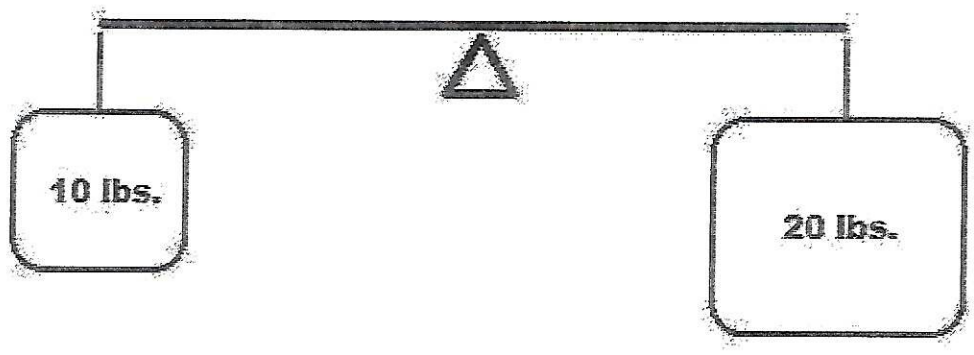

(2) If you have **unequal** weights at **equal** distances from the fulcrum, the scale is:

(a) tilted down on the side with the lighter weight

(b) balanced

(c) tilted down on the side with the heavier weight

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(3) If you have **equal** weights at **unequal** distances from the fulcrum, the scale is

(a) tilted down on the side with the weight closer to the fulcrum

(b) balanced

(c) tilted down on the side with the weight further from the fulcrum

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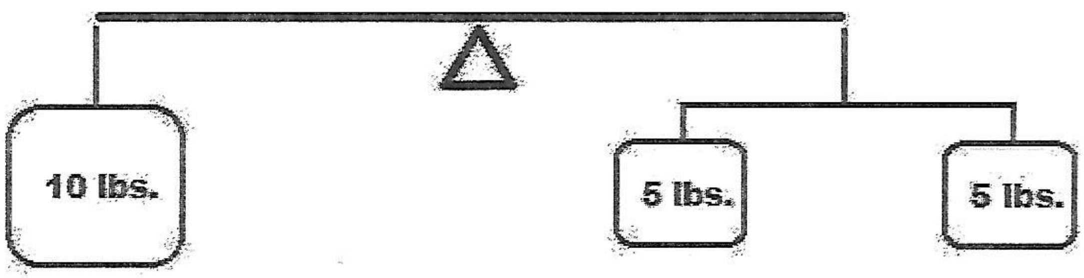


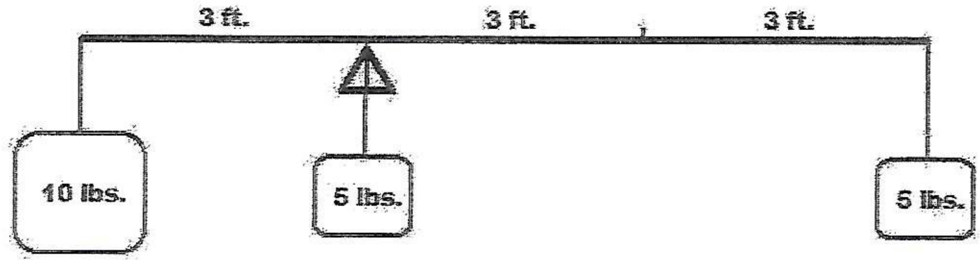
(4) The 5-lb. weights above are at equal distance from the cord their bar is hanging from (and that cord and the 10-lb. weight are at equal distances from the fulcrum). The scale will

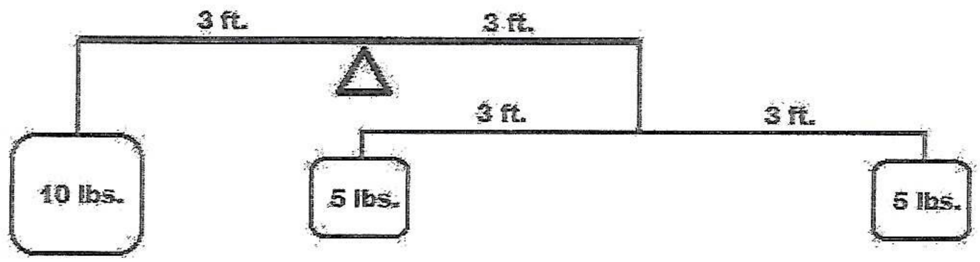
(a) tilt down on the left

(b) balance

(c) tilt down on the right

We can actually hang weights directly from the top bar and it doesn't make a difference—so the following two pictures are equivalent:





(5) In these configurations, the balance

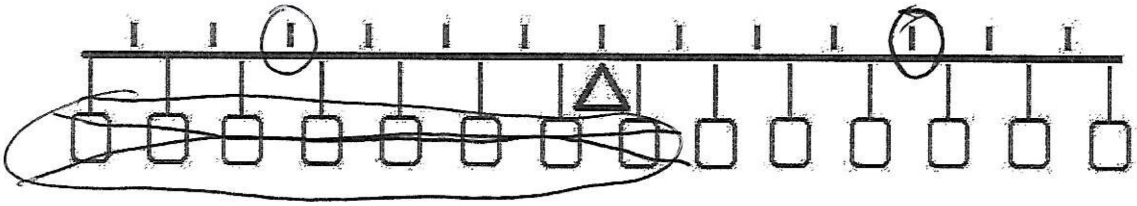
(a) tilts down on the left

(b) balances

(c) tilts down on the right

(6) What will happen if we remove the 5-lb. weight directly beneath the fulcrum?

Nothing because the weight beneath the fulcrum doesn't apply any torque anyways.



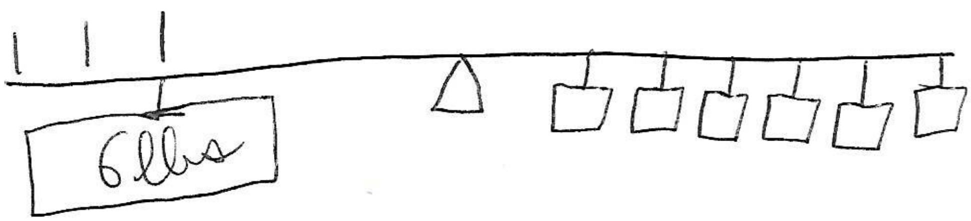
(7) The balance above has 14 small, 1-lb. weights suspended evenly across the top bar. (The marks on top are 1-foot apart—they are just there to help you measure distances.) Suppose you wanted to replace the **first 8** of them with a single large weight.

(a) How much should it weigh? 6 lbs

(b) Where should you place it? (How far from the fulcrum, on which side?)

Draw a picture of the new version, with the first 8 replaced. Label the distance and weight of the new weight:

Third indent from the left



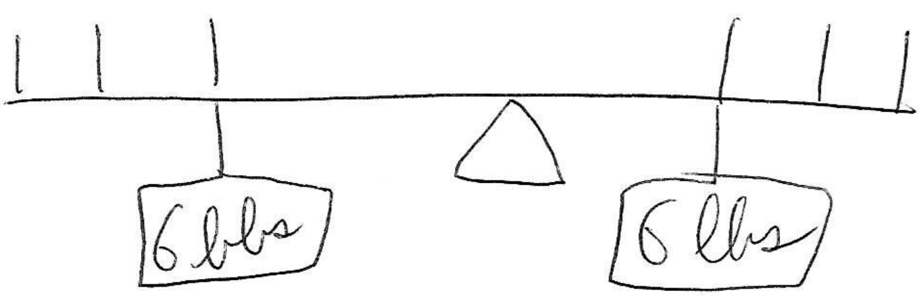
(8) Now that you've replaced the first 8 small weights with one big one, you'd like to replace the **last 6** of them with another single weight.

(a) How much should it weigh? 6 lbs

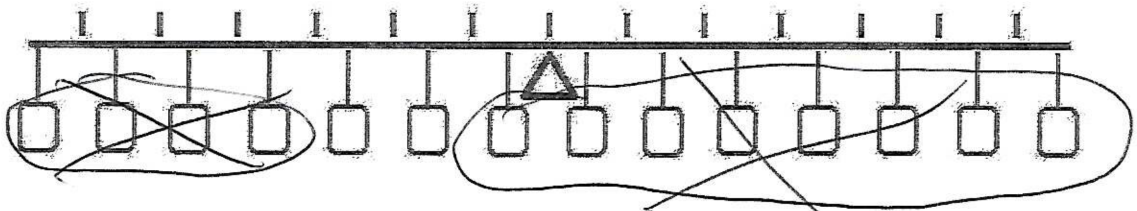
(b) Where should you place it?

Draw a picture of the new version, with both first 8 and last 6 replaced. Label the distances and weights of the two new weights:

The same spot on the opposite side

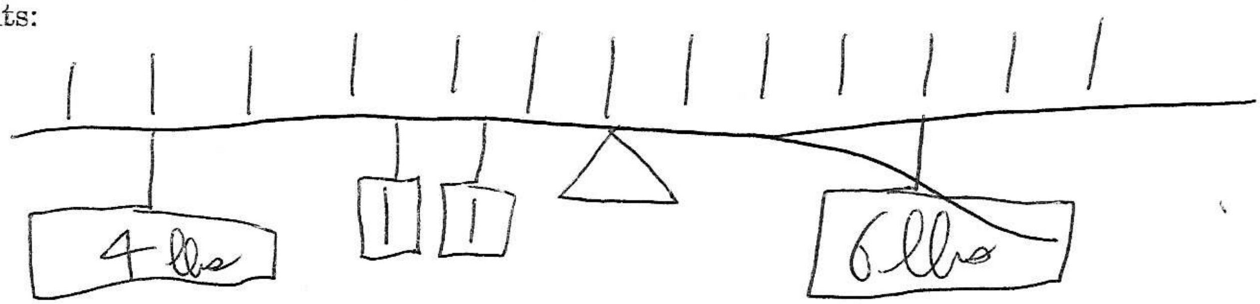


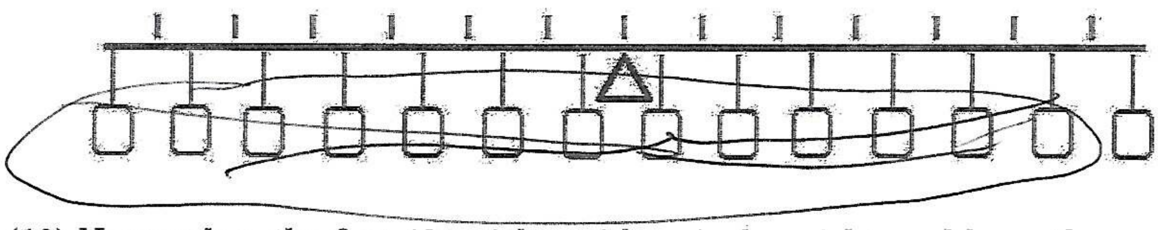
Here's the 14 small weights again:



(9) Now I want to replace the **first 4** small weights with one weight, and the **last 8** small weights with a second large weight.

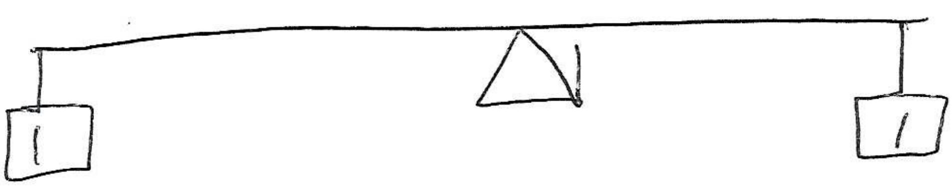
Draw a picture of the new version, labelling the distances and weights of the two new weights:





(10) Now replace the first 13 weights with a single weight, and leave the very last weight there.

Draw a picture of the new version, labelling the distances and weights of the two weights:



(11) In questions 7–10, would it have been possible to give different answers than the ones you did?

Only in (9), We could have put a 1 pound to replace the $\boxed{4}$ and then a $\boxed{3}$ to replace the $\boxed{6}$.

The Law of the Lever, first proved by the Greek mathematician Archimedes, describes the situation in which a scale/balance/lever will be balanced.

Consider a lever that has a weight of W_1, at a distance D_1 to the left of the fulcrum, and a weight of W_2 at distance D_2 to the right of the fulcrum (as pictured). Then the Law of the Lever states that the two weights will be balanced if, and only if

$$W_1 D_1 = W_2 D_2.$$

More generally, if there are multiple weights on each side, then the *sum* of the weight×distance values on the left side, must equal the sum of the weight×distance values on the right side.

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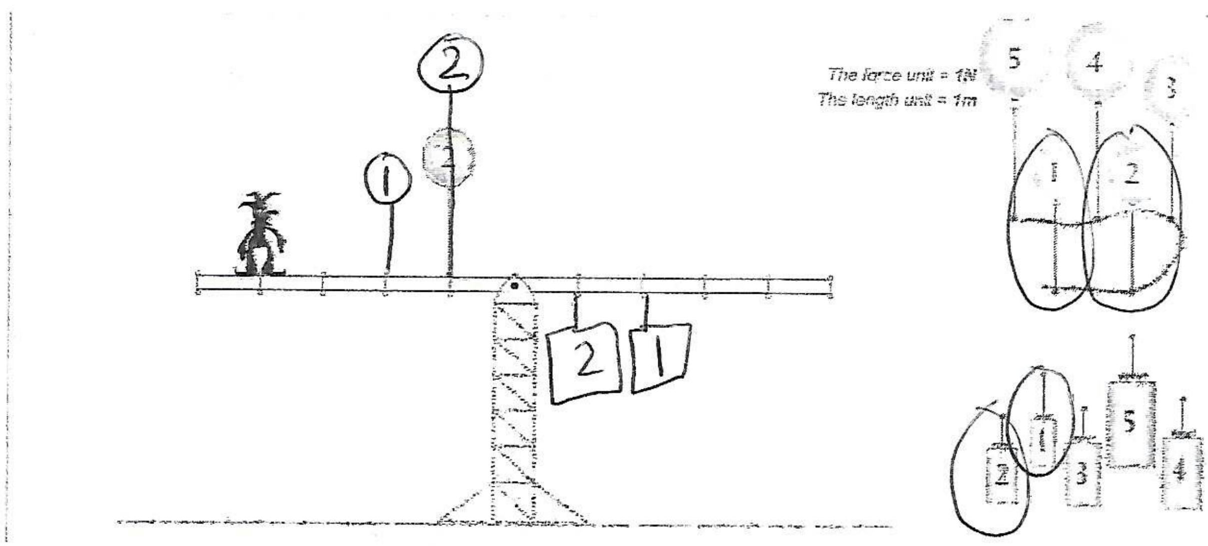
In the following pictures, your goal is to balance the beam. The jester weighs 1 unit of downward force, the weights have 1, 2, 3, 4, or 5 units of downward force, and the balloons weigh 1, 2, 3, 4, or 5 but they exert force *upward*.

For each picture, circle each balloon and weight which could be used to balance the beam if **only that weight/balloon** were attached to a single location on the bar that has a hook (note: hooks appear at distances of 1, 2, 3, 4, and 5 from the center on the left and right). Then draw at least one way that it can be done, using one of the weights/balloons you circled.



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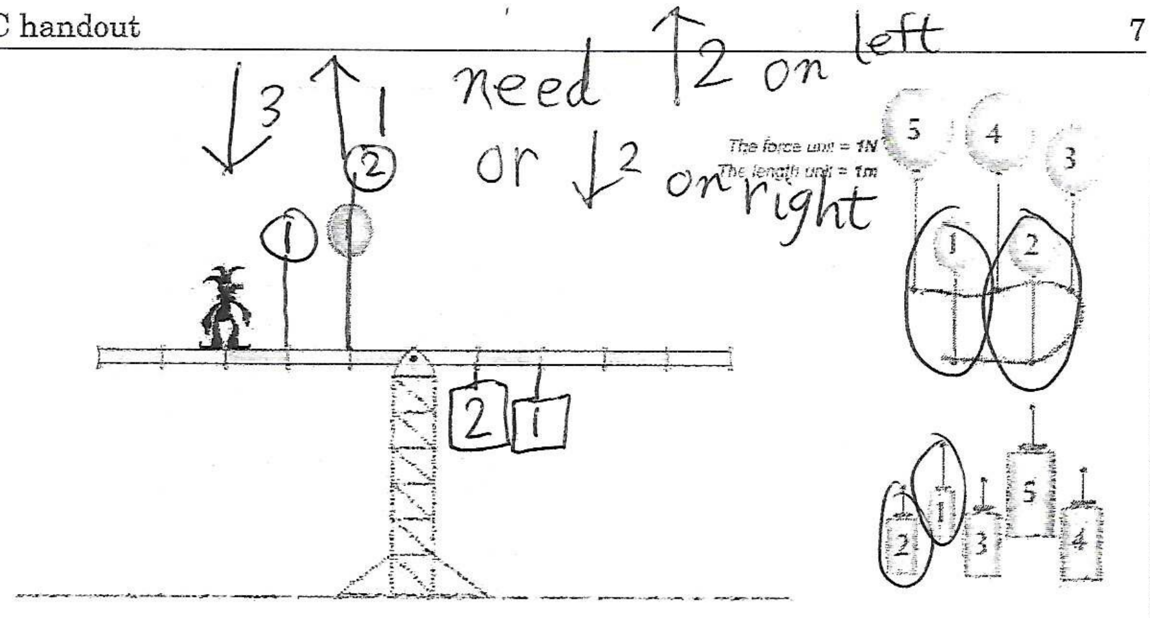
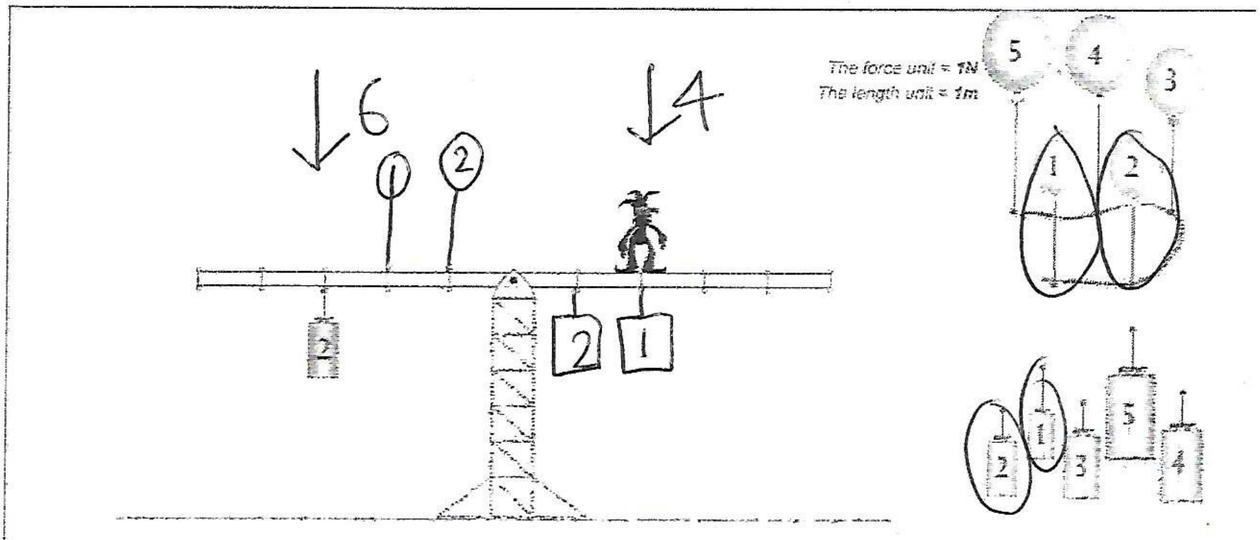


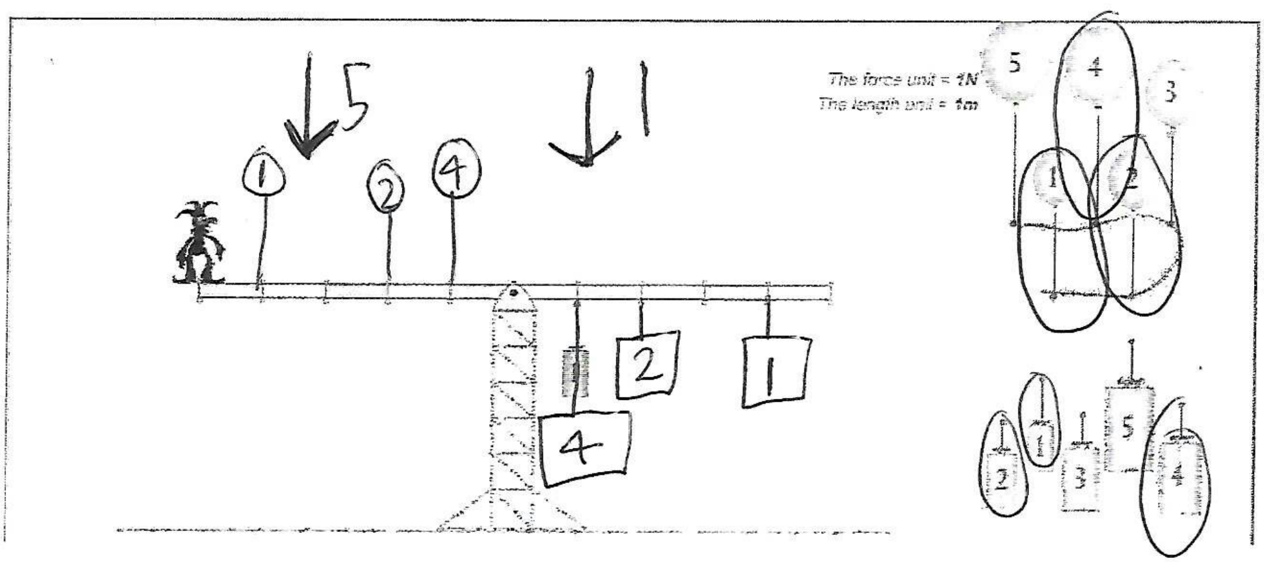
↓3 ↑1
need ↑2 on left
or ↓2 on right
The force unit = 1N
The length unit = 1m

=



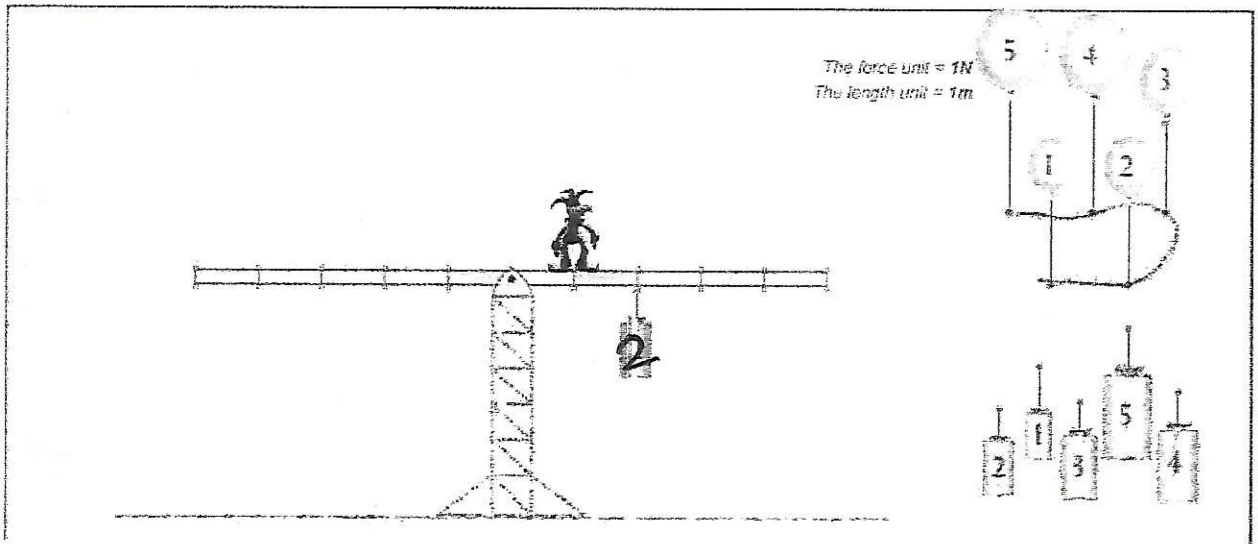
↓6
↓4
The force unit = 1N
The length unit = 1m

=

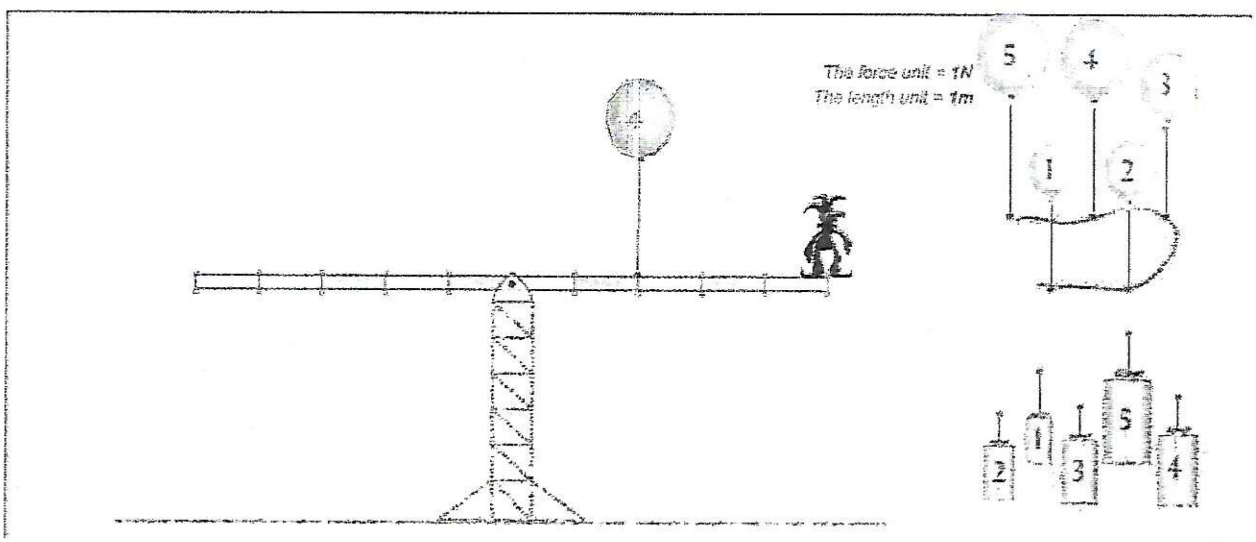


↓5
↓1
The force unit = 1N
The length unit = 1m

=


The force unit = 1N
The length unit = 1m

=


The force unit = 1N
The length unit = 1m

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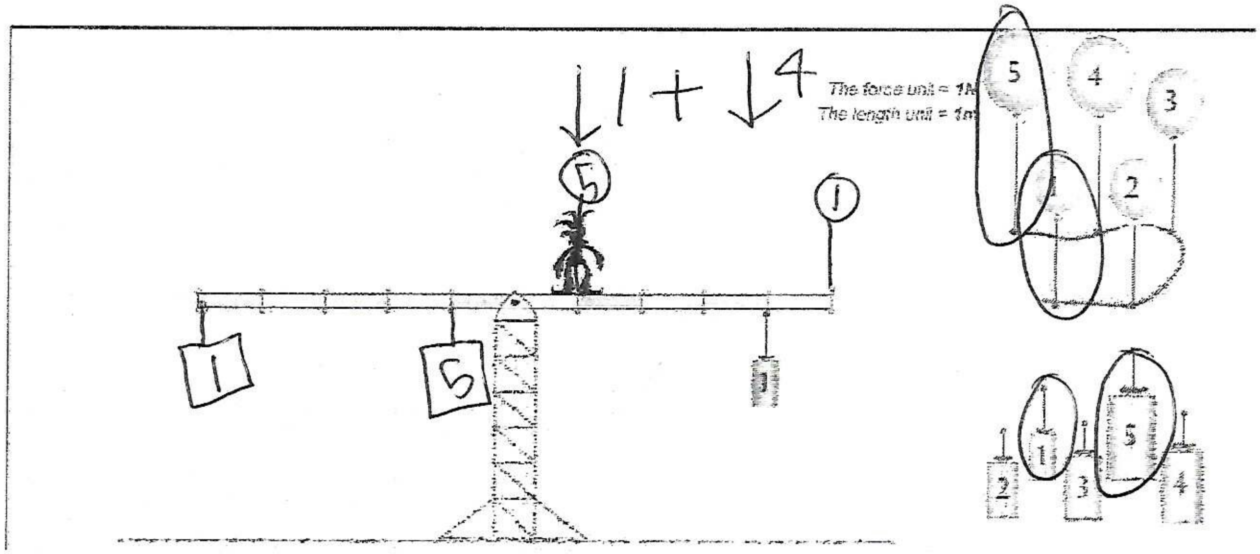
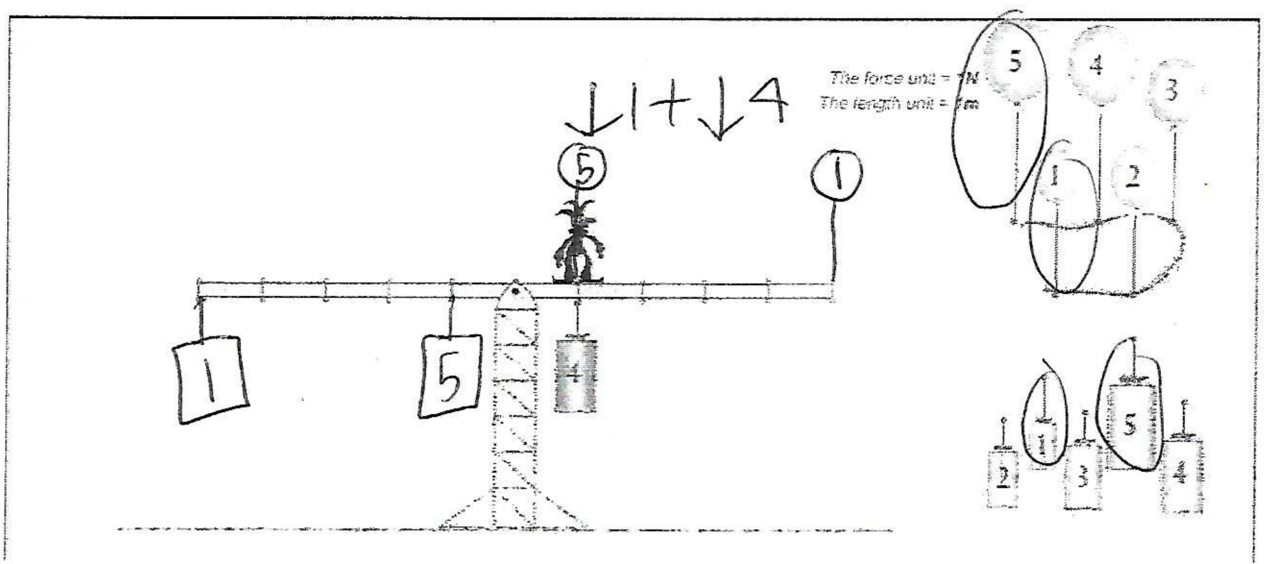


↓1 + ↓4
The force unit = 1N
The length unit = 1m

=



↓1+↓4
The force unit = 1N
The length unit = 1m

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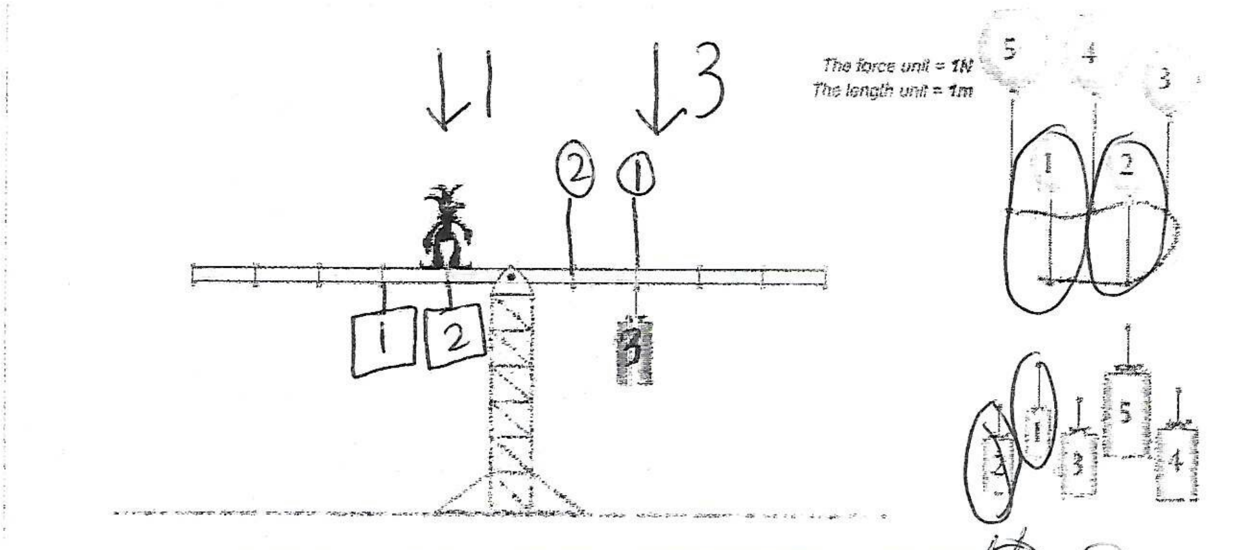
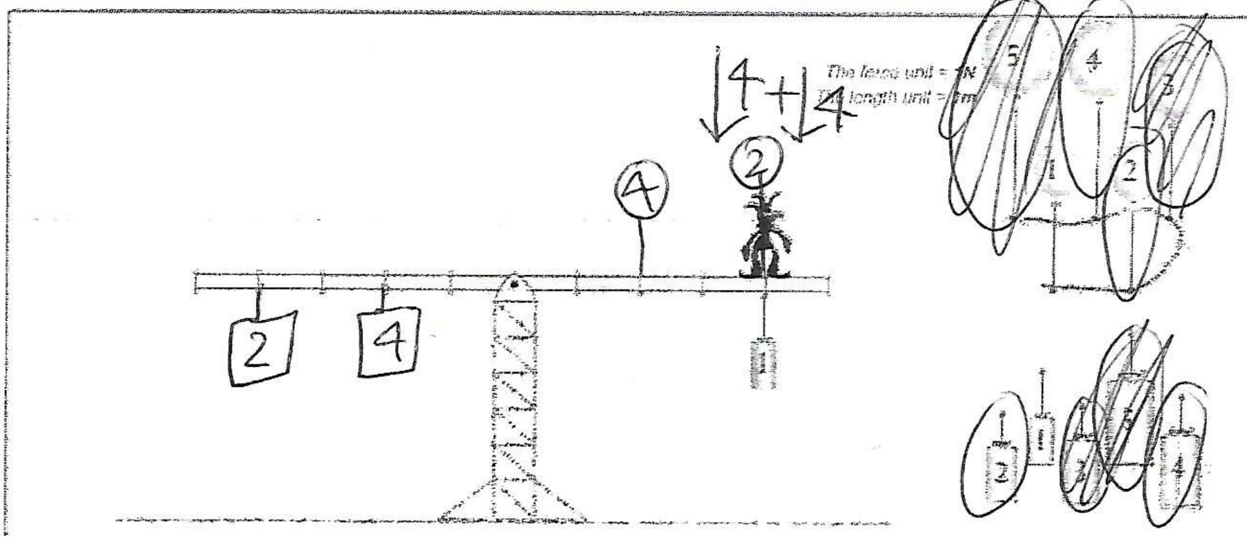
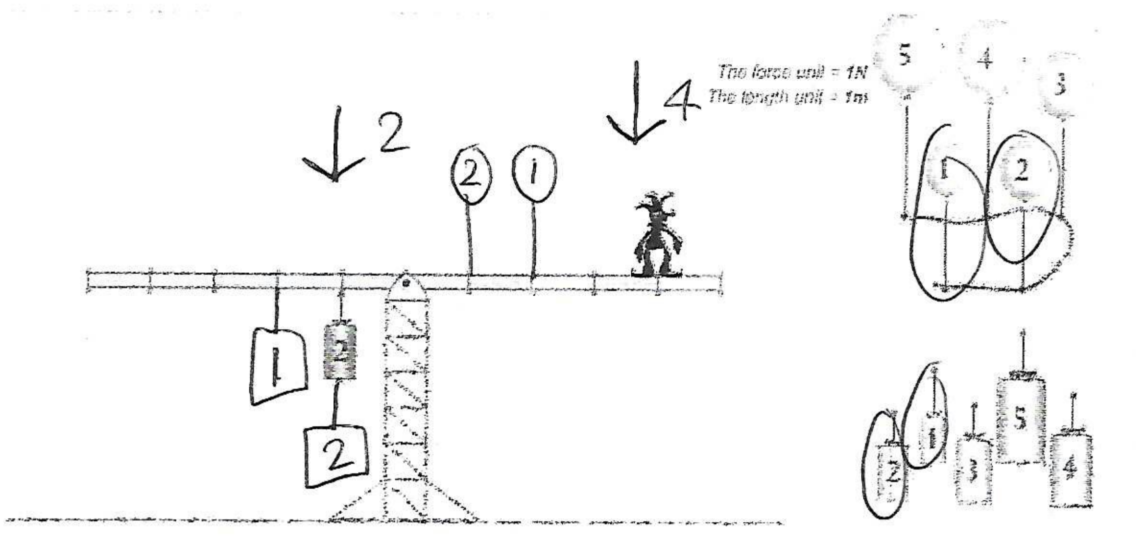

The force unit = 1N
The length unit = 1m


The force unit = 1N
The length unit = 1m


The force unit = 1N
The length unit = 1m

A LEVER PLAYING FIELD 2: NEVER SAY LEVER

MATH CIRCLE (BEGINNERS) 05/27/2012

The Law of the Lever, first proved by the Greek mathematician Archimedes, describes the situation in which a scale/balance/lever will be balanced.

Consider a lever that has a weight of W_1, at a distance D_1 to the left of the fulcrum, and a weight of W_2 at distance D_2 to the right of the fulcrum (as pictured). Then the Law of the Lever states that the two weights will be balanced if, and only if

$$W_1 D_1 = W_2 D_2.$$

More generally, if there are multiple weights on each side, then the *sum* of the weight$\times$distance values on the left side, must equal the sum of the weight$\times$distance values on the right side. For example if there were weights W_1 and W_2 on the left side at distances D_1 and D_2 respectively; and weight W_3 on the right side at distance D_3, then the scale will be balanced if and only if

$$W_1 D_1 + W_2 D_2 = W_3 D_3.$$

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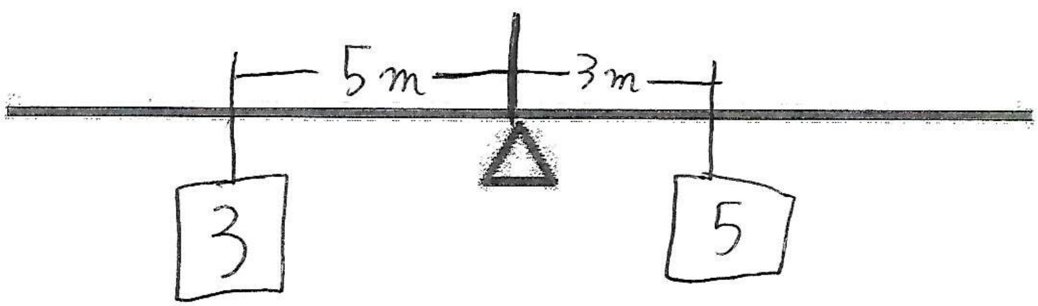
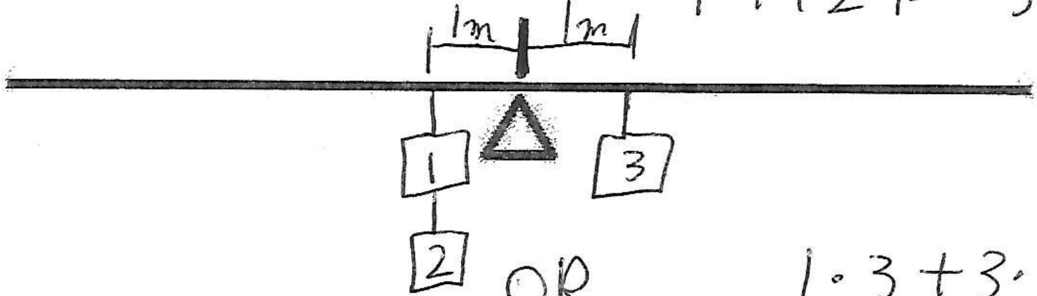
(1) You have a 3-lb. weight and a 5-lb. weight. On the lever below, place the weights so that the scale will be balanced. **In this and all subsequent problems, label the weights you use (here 3 lb, 5 lb), and indicate the distance of each one from the fulcrum.** (Hint: There is not just one possible answer, although there is arguably one that is "simplest".)



$3 \cdot 5 = 5 \cdot 3$

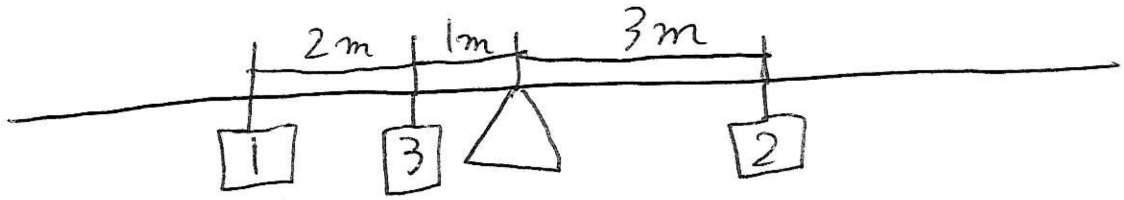
(2) You have a 1-lb. weight, a 2-lb. weight, and a 3-lb. weight. Place them on the scale so that it will be balanced, being sure to label the weights and their distances. (Again, there are multiple possible answers.)

$1\cdot 1+2\cdot 1=3\cdot 1$

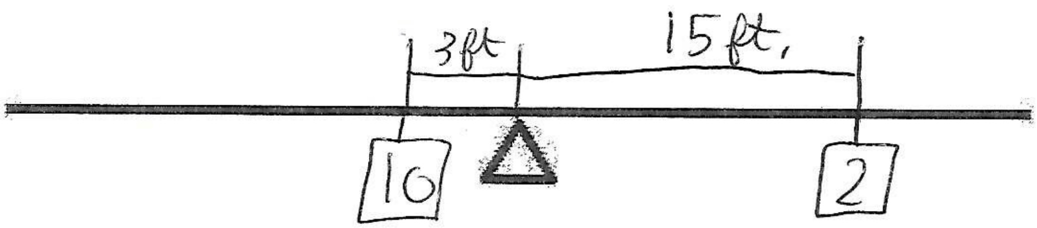


OR

$1\cdot 3+3\cdot 1=3\cdot 2$

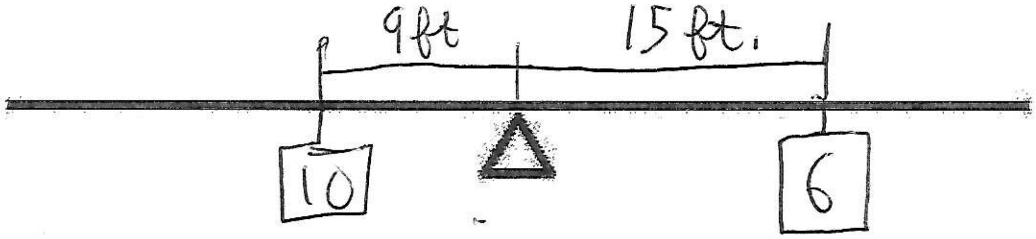


(3) Place a 10-lb. weight on the following scale, so that it balances with a weight which is 15 ft. from the fulcrum on the right side. You choose the weight on the right side, and the distance of the 10-lb. weight.



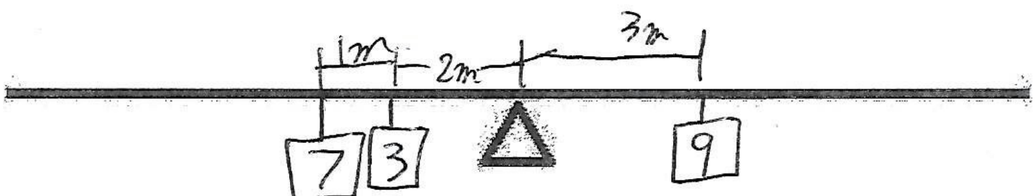
$10\cdot 3=2\cdot 15$

(4) Now solve problem (3) again, but give a different answer:



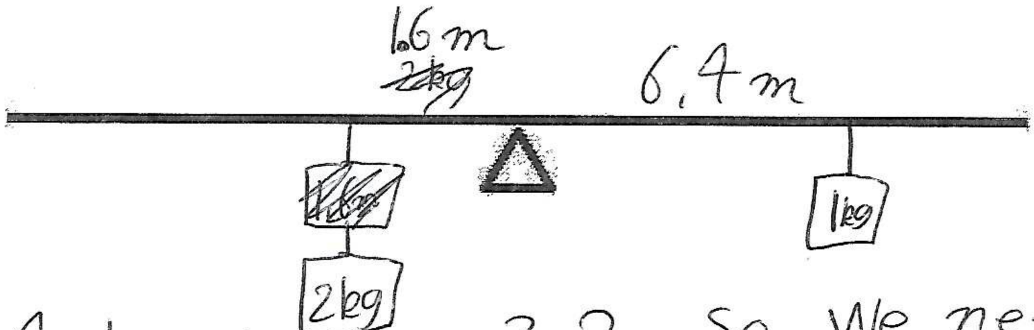
$10 \cdot 9 = 6 \cdot 15$

(5) This scale will use a 3kg weight, a 7kg weight, and a 9kg weight. The distances of the weights should be 2m, 3m, and 3m (but not necessarily in that order, and I'm not telling you on which side of the balance each distance should be!) Can you figure out where to place the weights, according to those rules, to balance the scale?



$3 \cdot 2 + 7 \cdot 3 = 6 + 21 = 27 = 9 \cdot 3 = 27$

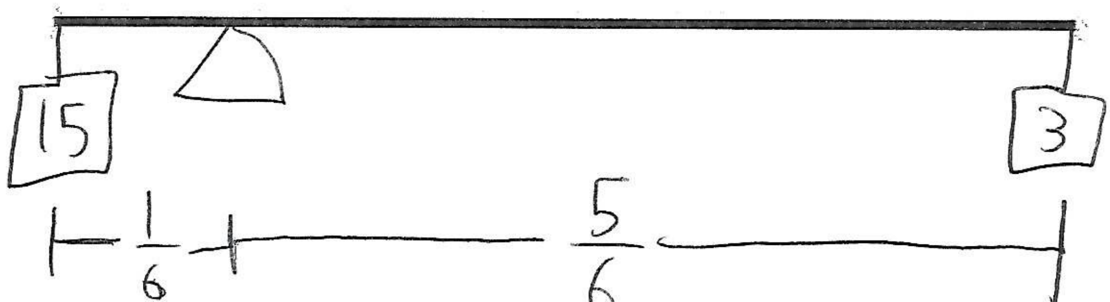
(6) The following balance is not balanced. It currently has a weight of 2kg suspended 1.6m to the left of the fulcrum, and a weight of 1kg suspended 6.4m to the right. **(Draw and label these weights and distances.)** The total length of the bar is 20m (10 on the left, 10 on the right). What is the weight of the smallest possible weight you could add to balance the scale, and where should you place it?



$6.4 \cdot 1 - 1.6 \cdot 2 = 3.2$, so we need 3.2 torque for weight on left edge.

$10 \cdot x = 3.2 \Rightarrow \boxed{x = 0.32\ kg}$

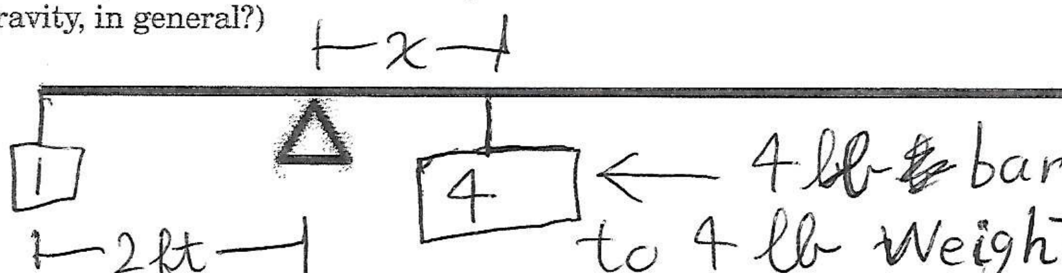
(7) The following bar has a weight of 15 lbs. at the far left and and 3 lbs. at the far right (draw them!). Where along the bar should you place the fulcrum so that it will balance? (Draw and label!) You can imagine that the bar itself weighs nothing.



$15 \cdot \frac{1}{6} = \frac{5}{6} \cdot 3 = \frac{15}{6}$ ✓

(8) In the previous problem you were supposed to imagine that the bar weighed nothing. Of course in reality a bar WILL weigh something. The bar below is actually rather heavy; it weighs 4 lbs. The distance from the fulcrum to the left end of the bar is 2 feet, and I have hung a 1 lb. weight at the left end. There is nothing hanging on the right side.

How long is the bar? (Picture may not be to scale.) (Hint: Where is a bar's center of gravity, in general?)



← 4 lb ~~lb~~ bar is equivalent to 4 lb weight hung at center

$1 \cdot 2 = 4x \Rightarrow x = 0.5$ ft,

half of bar is 2.5ft. so bar is $\boxed{5 \text{ ft.}}$

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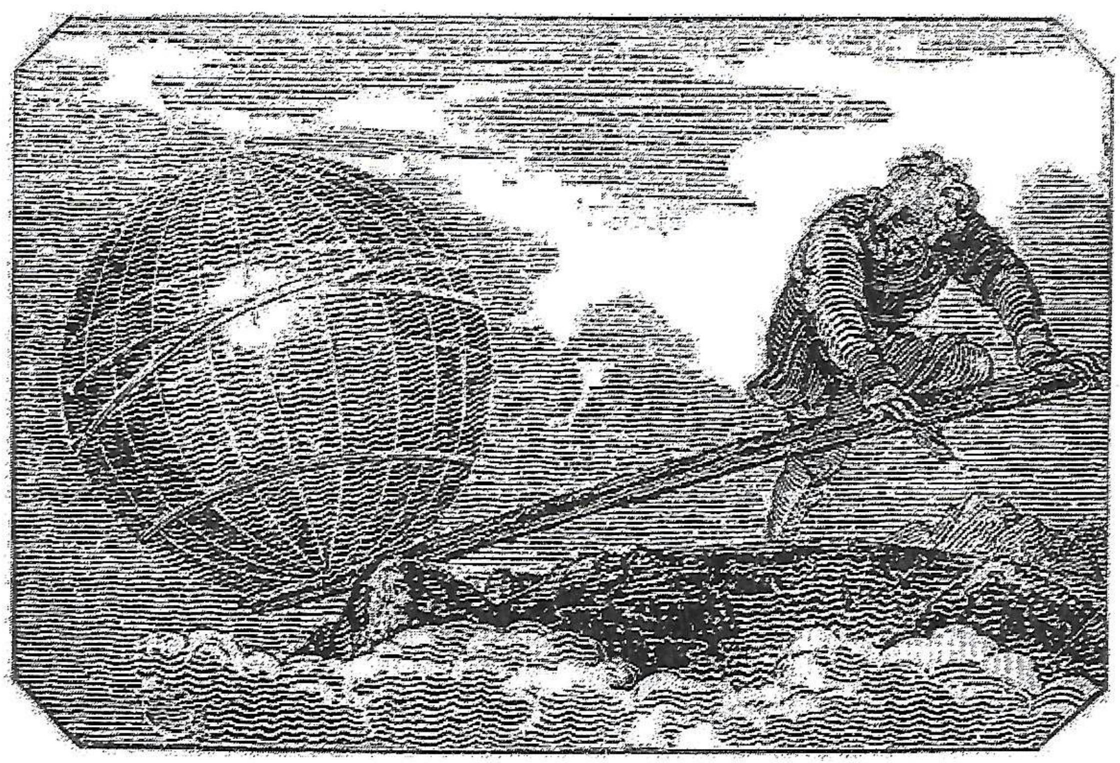
(9) Archimedes reportedly once boasted about the usefulness of levers, saying, "Give me the place to stand, and I shall move the earth." (See picture—not to scale.)



(a) About how much mass ("weight") does the Earth have, in either lbs. or kilograms? (Use any knowledge you, your neighbors, your assistant, etc. have to formulate a best-guess estimate.)

$6 \cdot 10^{24}$ kg

(b) About how many lbs. or kilograms (use same as above) of downward force can a human generate? (Use any knowledge you, your neighbors, your assistant, etc. have to formulate a best-guess estimate.)

~ 600 kg

(c) Given your answers to (a) and (b), about how long a lever would Archimedes need in order to move (or just balance) the world, assuming he found a place to stand, and the fulcrum were placed 1 meter from where the world rested on the lever? (You can imagine the lever itself is weightless, like in Problem (7).)

$\sim 10^{22}$ meters

(1 million light years!)