Combinatorics II

Get ready to get Deranged!

Math Circle

April 28, 2018

Before we move on with what I wanted to cover today, let us go back and revisit a couple of problems from last week's handout. If you already did them last week, great! They shouldn't take you much time to do again. If not, these problems are a great examples of combinatorics in action.

1. A recap of some problems from last week.

   (a) A *monotonic path* from $(0, 0)$ to (n, m) where $n, m \in \mathbb{N}$ is a path starting from $(0, 0)$ and ending at (n, m) which consists solely of moves up, or to the right by 1. Compute (combinatorially) the number of monotonic paths from $(0, 0)$ to (n, m).

   (b) Suppose that we have a monotonic path from $(0, 0)$ to (n, n). Define the exceedance of a path as the number of times that we move up, after we are already above the line $y = x$. Let P be a path with exceedance greater than one. Find a bijection between the paths with exceedance e and exceedance $e - 1$? This question's pretty hard, so if you can't get it after 10 minutes, ask an instructor for help!

(c) How many paths are there with exceedance 0? This number of called the n'th Catalan number.

(d) The parenthesis (()) and (()()) and ()()((((())()))) are all valid, but)(and ())(are not. A sequence S of (and)'s is called valid under two conditions. First, that there are the same number of ('s and)'s. What is the second condition?

(e) Before starting this question, check to make sure with an instructor to make sure that you got the last one right. How many valid strings of $2n$ parenthesis are there? Prove it combinatorially.

2. Now let's talk about a nice theorem, called the principle of inclusion/exclusion. This principle give you a useful way of rewriting one complicated statement into a bunch of simpler ones.

   (a) For a set $\mathcal{A}$, $|\mathcal{A}|$ is the number of elements in $\mathcal{A}$. Prove that $|\mathcal{A} \cup \mathcal{B}| = |\mathcal{A}| + |\mathcal{B}| - |\mathcal{A} \cap \mathcal{B}|$? After you can prove it, how would you explain this principle for a non-mathematician in a sentence or two?

   (b) What about if we have more than two sets? Rewrite $|\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}|$ as much as possible by applying the principle of inclusion/exclusion. Check with an instructor once you answer this problem.

   (c) Alice, Bob, Charlie, Diana, Ernest and Francesca are all in line to to buy some bubblegum/caramel flavored bagels. How many ways can they arrange themselves so that no three consecutive alphabetically-increasing people are next to each other in line? In other words, Alice, Bob and Charlie can't be the first three, nor 2 - 4, 3 - 5 or 4 - 6.

(d) AMC 10B problem 13. There are 20 students participating in an after school program offering classes in yoga, shop, and extreme painting. Each student must take at least one of the classes, but may take more than one. There are 10 students taking yoga, 13 shop and 9 painting. There are 9 students taking at least 2 of the classes. How many are taking all 3?

(e) There are some problems in math that are really deranged. Specifically, a bijection ω from $\mathcal{A}$ to itself is called a derangement if ω has no fixed points (in other words, there are **no** $x \in \mathcal{A}$ such that $\omega(x) = x$). Let $[n] = \{1, 2, \ldots, n\}$. How many derangements are there on [1], [2] and [3]?

(f) Can you come up with a combinatorial proof that $D(n) = (n-1)(D(n-1) + D(n-2))$?

(g) **Feeling saucy? Prove that $D(n) = \sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i} i!$. A good starting point might be verifying that this formula satisfies the above problem. Once that is done, try and find a combinatorial proof.**

(h) **It is (an extremely surprising) fact that $D(n) \approx \frac{n!}{e}$. Can you prove this? It's not as hard as you might think, but the proof that I know relies on some calculus knowledge.**

If there is time at the end, we'll watch the short (for real this time) video from numberphile called 'Derangements - Numberphile'

`https://www.youtube.com/watch?v=pbXg5EI5t4c`

3. Lastly, here are some bonus challenge problems. These problems are again taken from Enumerative combinatorics by Prof. Richard Stanley.

(a) **Try and prove this number theoretic fact combinatorially. Let p be prime, and a be an integer. Show that $a^p - a$ is divisible by p combinatorially. Specifically, construct a set S with $a^p - a$ elements and partition S into p disjoint subsets so that each subset has an equal number of elements.**

(b) **Let p be prime and let $n = \sum a_i p^i$ and $m = \sum b_i p^i$ by the p-ary expansions of the positive integers m and n. Show that $\binom{n}{m} = \binom{a_0}{b_0}\binom{a_1}{b_1}\ldots \bmod(p)$.**

(c) **Use the previous problem to determine when $\binom{n}{m}$ is odd. For what n is $\binom{n}{m}$ odd for all $0 \le m \le n$? In general, how many coefficients of the polynomial $(1+x)^n$ are odd?**

(d) **It then follows, and is easy to show directly, that $\binom{pa}{pb} \equiv \binom{a}{b} \bmod(p)$. Give a combinatorial proof that $\binom{pa}{pb} \equiv \binom{a}{b} \bmod(p^2)$**