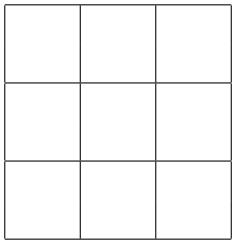
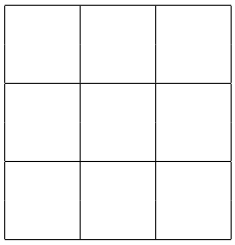
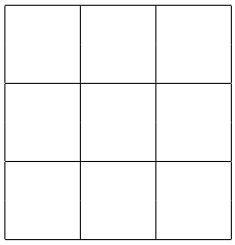
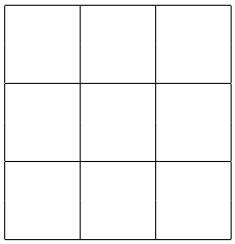
1. On the following 3×3 chessboard there are black knights in the top two corners and white knights at the bottom two corners.

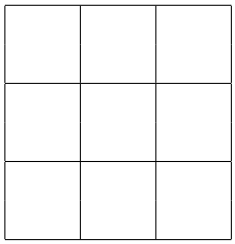


The knights are allowed to move as knights normally do in chess, but they do not capture one another or occupy the same space. Which of the following arrangements are possible to achieve after several legal moves:









2. In the $24\frac{1}{2}$th century, there are daily rockets traveling both ways on the following routes:

   - Earth-Mercury
   - Moon-Venus
   - Earth-Moon
   - Moon-Mercury
   - Mercury-Venus
   - Uranus-Neptune
   - Neptune-Saturn
   - Saturn-Jupiter
   - Jupiter-Mars
   - Mars-Uranus

   Is it possible to get from Earth to Mars?

3. Shortly after the $24\frac{1}{2}$th century, rocket routes are established between every pair of planets, and between every planet and the moon. How many rocket routes are there?

4. A party is held with 15 people, and handshaking ensues. Prove that at the end of the party, two people have shaken the same number of hands.

Graphs

It is possible to solve each the previous problems with the aid of a graph. We say that a **graph** is a collection of **vertices**, with some pairs of vertices connected by **edges**. We usually represent graphs as a collection of dots connected by lines:

It is common to use the letters V and E to denote the number of vertices and the number of edges in a graph.

Graph problems

1. Which of the following pairs of graphs are "the same"? What do you think it should mean for two graphs to be "the same"?

2. A graph has 15 vertices, and each vertex has at least 7 edges emerging from it. Prove that there is a path of edges connecting any vertex to any other vertex.

3. A graph is called a **planar graph** if it can be drawn in a plane such that none of its edges cross. We say that a planar graph is **properly drawn** if it is drawn without edges crossing. Which of these graphs are planar? (Note that they may not be properly drawn!)

When a planar graph is drawn properly, it separates the plane into some number of regions, or **faces**. In addition to the notations V and E, it is common to denote the number of faces by F. For each graph, draw it properly if it is not already drawn that way, and then count the number of vertices, edges, and faces.

4. (a) Prove that for any planar graph, $2E \geq 3F$.

   (b) Prove that if there are no trios of vertices all connected to one another, then $E \geq 2F$.