Graph Theory for Middle School Students

Los Angeles Math Circle

Olga Radko
radko@math.ucla.edu

Oleg Gleizer
ogleizer@geffenacademy.ucla.edu

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2 Copyright and acknowledgements

O. Gleizer would like to thank his friend Dr. Benny Sudakov for an introduction to Ramsey theory.

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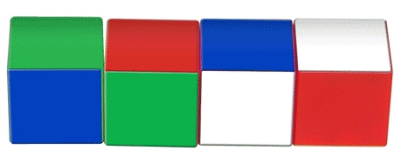
3 Instant Insanity

Instant Insanity is a popular puzzle, created by Franz Owen Armbruster, currently marketed by the *Winning Moves* company, and sold, among other places, on *Amazon.com.* It is advisable to have the puzzle in front of you before reading this chapter any further.

The puzzle consists of four cubes with faces colored with four colors, typically red, blue, green, and white. The objective of the puzzle is to stack the cubes in a row so that each side, front, back, upper, and lower, of the stack shows each of the four colors.



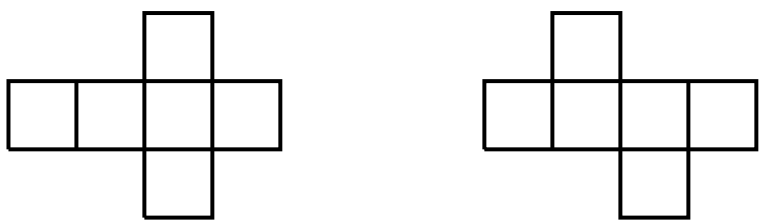
There exist 41,472 different arrangements of the cubes. Only one is a solution. Finding this one by trial and error seems about as likely as winning a lottery jackpot. However, we have witnessed a few LAMC students doing just that. Those were some truly extraordinary children!

Problem 1 *Try to solve the puzzle.*

To approach a task this formidable, the more ordinary people, like the authors of this book, need to forge some tools.

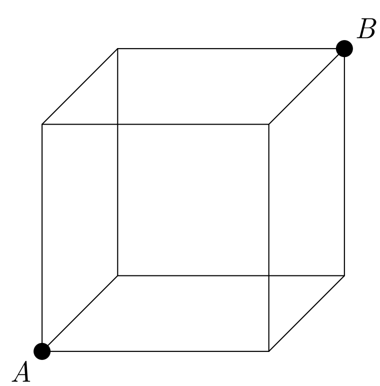
Cubic nets

A *cubic net* is a 2D picture simultaneously showing all the six sides (a.k.a. faces) of a 3D cube, please take a look at the examples below.



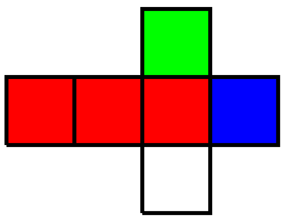
Problem 2 *Draw a cubic net different from the two above.*

Problem 3 *An ant wants to crawl from point A of a cubic room to the opposite point B, please see the picture below.*

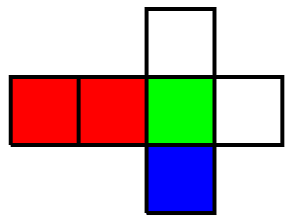


The insect can crawl on any surface, a floor, ceiling, or wall, but cannot fly through the air. Find at least two different shortest paths for the ant (there is more than one). Hint: use a cubic net.

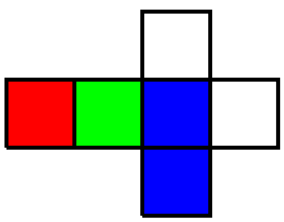
Now we have the means to take a better look at the cubes from the puzzle, the cubic nets!



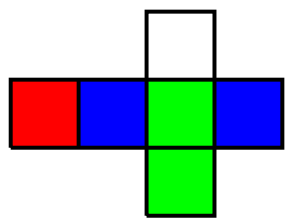
Cube 1



Cube 2



Cube 3



Cube 4

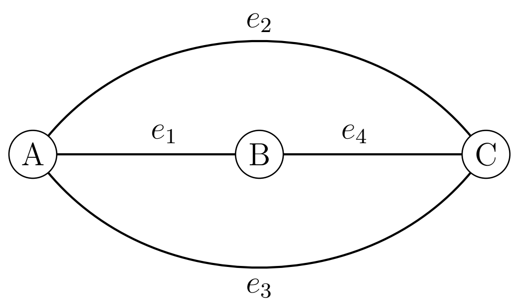
We can see that all the four cubes of the puzzle are different. Cube 1 is the only one having three red faces. Cube 2 uniquely possesses two red faces. Cube 3 is the only one having two adjacent blue faces. Cube 4 also has two blue faces, but they are opposite to each other. Finally, Cube 4 is the only one having two green faces.

Cubic nets are great for visualizing a single cube, but they are not as efficient at describing various configurations of all the four of them. We need one more tool.

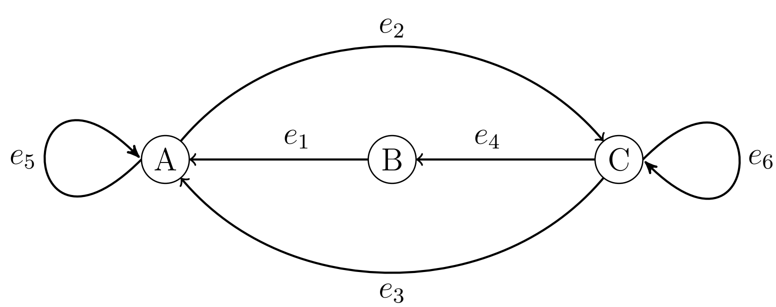
Graphs

A *graph* is a set of vertices, $\mathcal{V} = \{v_1, v_2, \ldots\}$, connected by edges, $\mathcal{E} = \{e_1, e_2, \ldots\}$. If an edge e connects the vertices v_i and v_j, then we write

$e = \{v_i, v_j\}$. If the order of the vertices does not matter, the graph is called *undirected*. Typically, the word *graph* means an undirected graph. A graph is called a *directed graph*, or a *digraph*, if the order of the vertices does matter. For example, the (undirected) graph below has three vertices, A, B, and C, and four edges, $e_1 = \{A, B\}$, $e_2 = \{A, C\}$, $e_3 = \{A, C\}$, and $e_4 = \{B, C\}$.



An edge connecting a vertex to itself is called a *loop*. For example, the digraph below has two loops, $e_5 = (A, A)$ and $e_6 = (C, C)$, in addition to the edges $e_1 = (B, A)$, $e_2 = (A, C)$, $e_3 = (C, A)$, and $e_4 = (C, B)$.



Note that we use different notations for an edge of a graph and digraph. An edge of a graph, $e = \{A, B\}$, is a set of the two vertices it connects. In this case, the order does not matter, $\{A, B\} = \{B, A\}$ as sets. An edge of a digraph, $e = (A, B)$ is a list (an ordered set) of the vertices it connects. The order does matter now, $(A, B) \neq (B, A)$.

The endpoint of a directed edge e is called its *head* and denoted $h(e)$. The starting point on an edge e is called its *tail* and denoted $t(e)$. For example,

$h(e_4) = B$ and $t(e_4) = C$ for the digraph above.

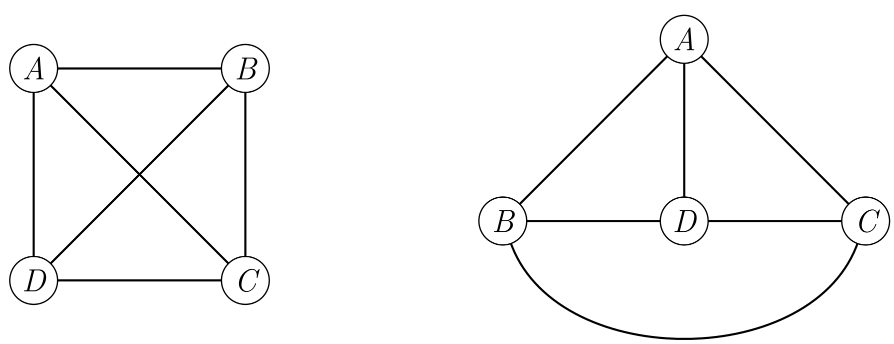
In the book, we use the letters $\mathcal{V}$ and $\mathcal{E}$ for the sets of vertices and edges in a graph. We use the letters V and E for the numbers of the vertices and edges. In other words, V is the number of elements in the set $\mathcal{V}$, E is the number of elements in the set $\mathcal{E}$.

Problem 4 *Given a graph with V vertices and E edges that has no loops, how many ways are there to orient the edges so that the resulting digraphs are all different?*

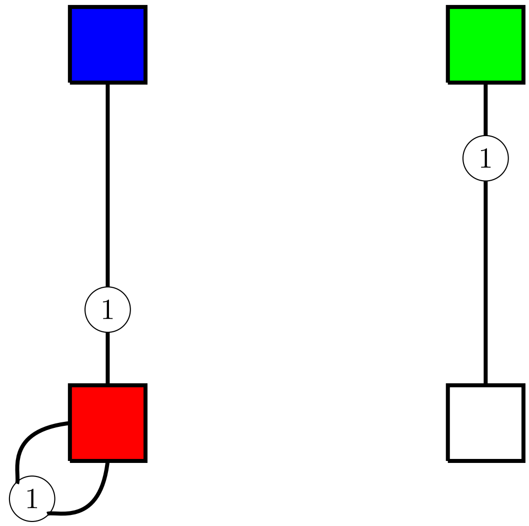
Problem 5 *Draw an undirected graph that has the vertices A, B, C, D, and E and the edges $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{A, E\}$, $\{B, C\}$, $\{C, D\}$, and $\{D, E\}$.*

Two different pictures of a graph can look very dissimilar.

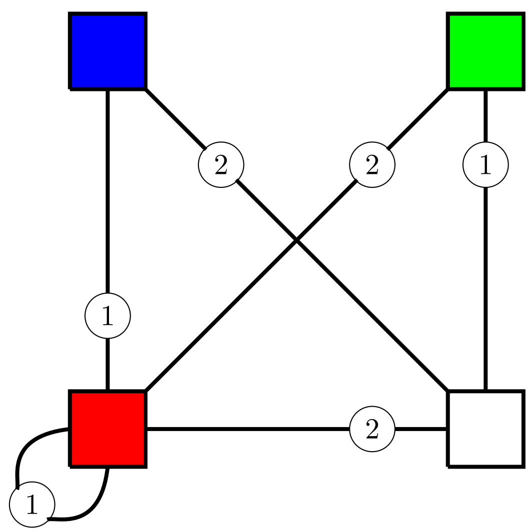
Problem 6 *Prove that the two pictures below represent the same graph by comparing the sets of their vertices and edges.*



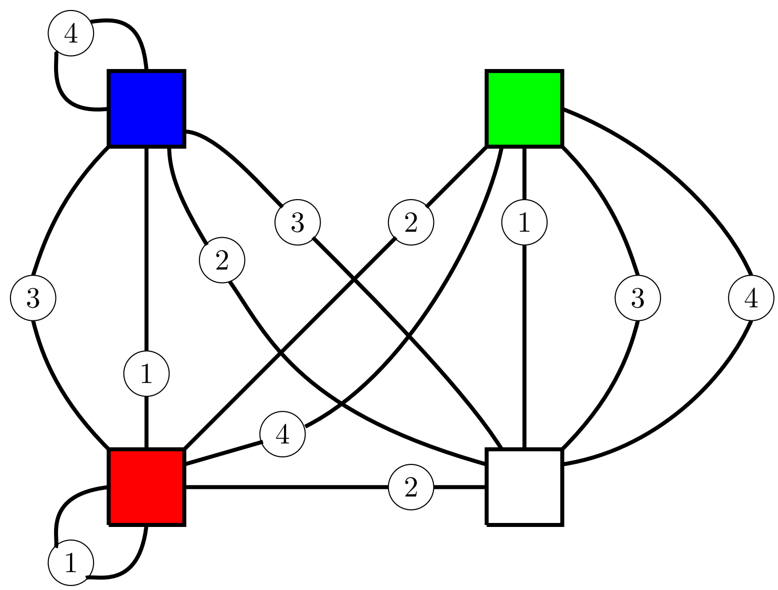
Getting back to the puzzle, let us represent Cube 1, see page 4, by a graph. The vertices will be the face colors, Blue, Green, Red, and White, $\mathcal{V} = \{B, G, R, W\}$. Two vertices will be connected by an edge if and only if the corresponding faces are opposing each other on the cube. Cube 1 has the following edges, $e_1 = \{B, R\}$, $e_2 = \{G, W\}$, and the loop $e_3 = \{R, R\}$. To emphasize that all the three edges represent the first cube, let us mark them with the number 1.



Cube 2 has the following pairs of opposing faces, $\{B, W\}$, $\{G, R\}$, and $\{R, W\}$. Let us add them to the graph as the edges e_4, e_5, and e_6.



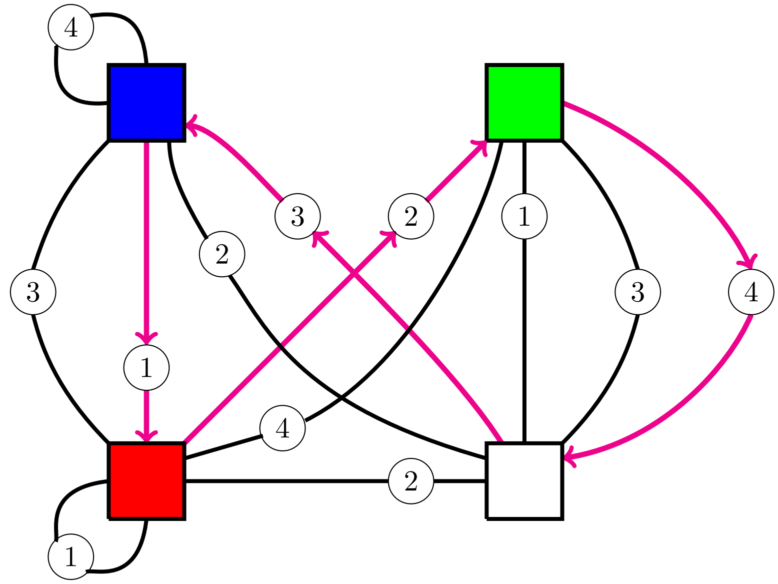
Let us now make the graph represent all the four cubes.



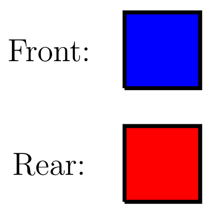
Problem 7 *Check if the above representation is correct for Cubes 3 and 4.*

With the help of the above graph, solving the puzzle becomes as easy as a walk in the park, literally. Imagine that the vertices of the above graph are the clearings and the edges are the paths. An edge marked by the number i represents two opposing faces of the i-th cube. Let us try to find a closed walk, a.k.a. a cycle, in the graph that visits each clearing once and uses the paths marked by the different numbers, $i = 1, 2, 3, 4$. If we order the front and rear sides of the cubes accordingly, then the front and rear of the stack will show all the four different colors in the order prescribed by our walk.

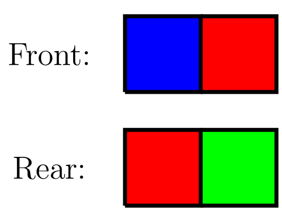
For example, here is such an (oriented) cycle, represented by the magenta arrows on the picture below.



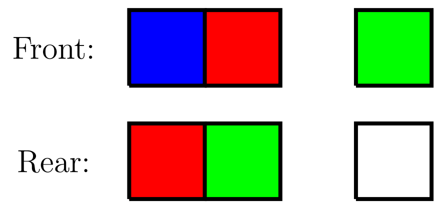
The first leg of the walk tells us to take Cube 1 and to make sure that its blue side is facing forward. Then the red side, opposite to the blue one, will face the rear.



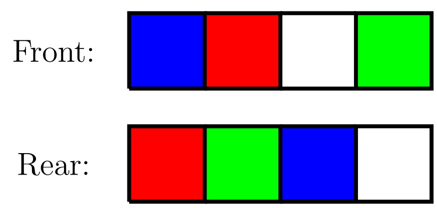
The next leg of the walk tells us to take Cube 2 and to place it in such a way that its red side faces us while the opposing green side faces the rear. Since we go in a cycle that visits all the colors one-by-one, neither color repeats the ones already used on their sides of the stack.



The third leg of the walk tells us to take Cube 4, not Cube 3, and to place it green side forward, white side facing the rear.



Finally, the last leg of the walk tells us to take Cube 3 and to place it the white side facing forward, the opposite blue side facing the rear.



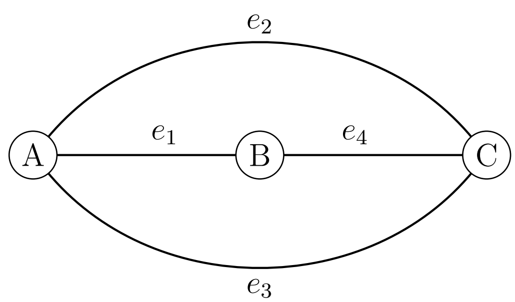
Now the front and rear of the stack are done. If we manage to find a second oriented cycle in the original graph that has all the properties of the first cycle, but uses none of its edges, we would be able to do the upper and lower sides of the stack and to complete the puzzle. Using the edges we have already traversed during our first walk will mess up the front-rear configuration, but there are still a plenty of the edges left!

Problem 8 *Complete the puzzle.*

4 Elementary properties of graphs

Two vertices of a graph are called *adjacent*, if they are connected by an edge. Two edges of a graph are called *incident*, if they share a vertex. Also, a vertex and an edge are called *incident*, if the vertex is one of the two the edge connects.

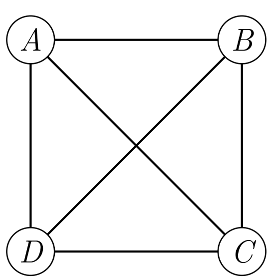
Consider a graph with m vertices, $v_1, v_2, \ldots, v_m$ and n edges, $e_1, e_2, \ldots, e_n$. The *incidence matrix* of the graph is an $m \times n$ table of numbers T organized the following way. In the case the vertex v_i is incident to the edge e_j that is not a loop, $T_{ij} = 1$. If e_j is a loop, $T_{ij} = 2$. $T_{ij} = 0$ otherwise. For example, on the right hand side below is the incidence matrix of the first graph on page 5, reproduced for your convenience below on the left hand side.

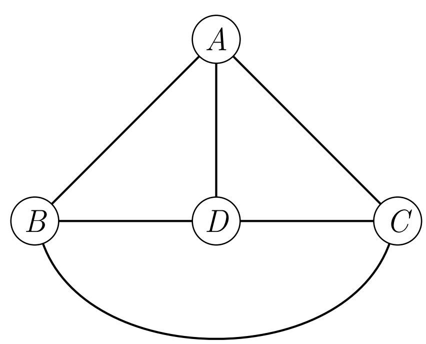


$$T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Problem 9 *Looking at the incidence matrix T above, it is not hard to notice that the sum of the entries in every column equals 2. Would it always be the case for an undirected graph? Why or why not?*

Problem 10 *Write down the incidence matrices T_1 and T_2 for the graphs from Problem 6, reproduced for your convenience below. Do the matrices tell you that you are looking at two different pictures of one and the same graph? Why or why not?*

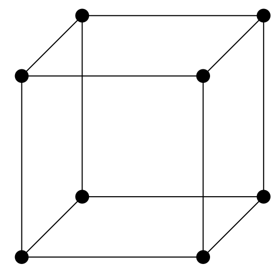




Problem 11 *Write down the incidence matrix of the Instant Insanity configurations' graph on page 8.*

The *degree* $d(v)$ of a vertex v of a graph is the number of the edges of the graph incident to the vertex.

Problem 12 *Do all the vertices of a cube, considered as a graph, have the same degree? If so, what is the degree?*



Theorem 1 *For any graph, the sum of the degrees of the vertices equals twice the number of the edges.*

Problem 13 *Prove Theorem 1.*

Problem 14 *Prove the following corollary of Theorem 1. The number of vertices of odd degree in any graph is even.*

Problem 15 *One girl tells another, "There are 25 kids in my class. Isn't it funny that each of them has 5 friends in the class?" "This cannot be true," immediately replies the other girl. How did she know?*

Problem 16 *In a small European country, each city is connected to other cities of the country by five roads. There are 25 inter-city roads in the country. How many cities are there?*

A *path* in a graph is a subgraph having the following property. Its vertices can be renumbered $v_1, v_2, \ldots, v_n, v_{n+1}$ so that $e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, \ldots, e_n = \{v_n, v_{n+1}\}$. In other words, a path is a subgraph that can be drawn without lifting the pen off the paper. A *cycle* is a closed path, i.e. a path such that $v_1 = v_{n+1}$.

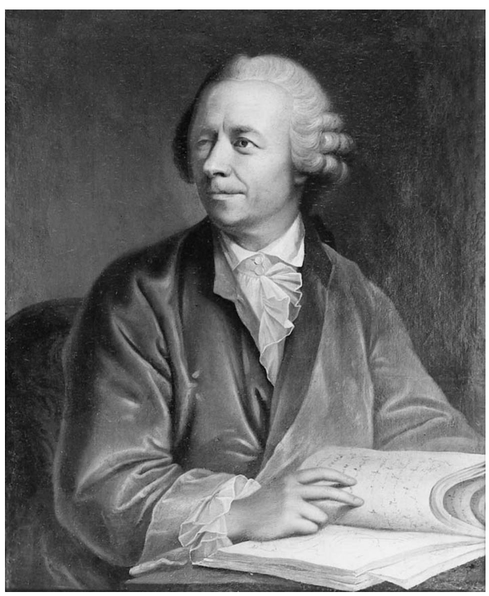
A graph is called *connected* when there is a path between every pair of its vertices. A graph is called *disconnected* otherwise.

Problem 17 *Draw a graph with four vertices, all of degree one.*

Problem 18 *By the year 2050, Hyperloop has developed the following routes: Portland – New York City, New York City – Boston, Boston – Los Angeles, Los Angeles – San Francisco, San Francisco – New York City, San Francisco – Portland, San Diego – Washington DC, San Diego – Austin, Austin – Charlotte, Charlotte – Washington DC, Austin – Washington DC, and Los Angeles – New York City. Each route has high-speed passenger and cargo pods travelling both ways. Is it possible to get from Los Angeles to Washington DC by a Hyperloop pod? Why or why not?*

5 Eulerian paths and cycles

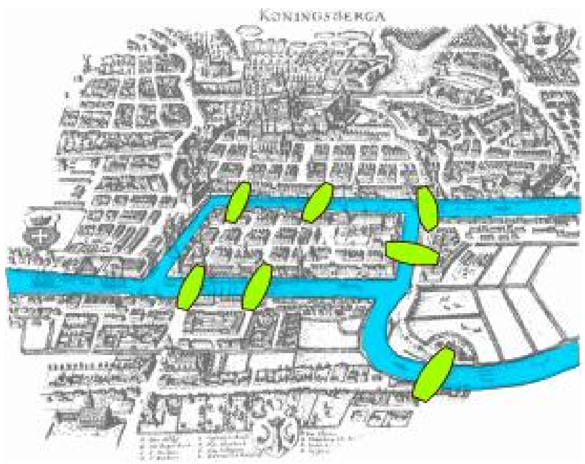
An *Eulerian path* is a path in an (undirected) graph that traverses each edge exactly once. An *Eulerian cycle* is a closed Eulerian path. They are named in honour of a great Swiss mathematician, Leonhard Euler (1707-1783), considered by many as the founding father of the graph theory.



Leonhard Euler

The seven bridges of Königsberg problem

During his stay in the city of Königsberg, then the capital of Prussia, Euler came up with, and solved, the following problem. Can one design a walk that crosses each of the Königsberg's seven bridges once and only once? The picture of Königsberg of Euler's time is provided below.

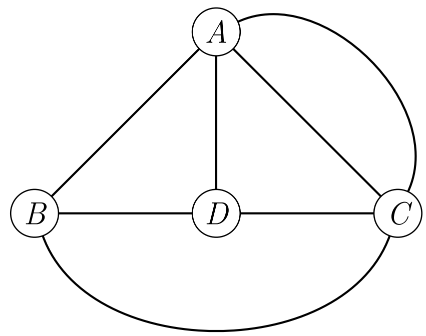


Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges.

Problem 19 *Draw a graph with the vertices corresponding to the landmasses from the picture above and with the edges corresponding to the Königsberg's seven bridges. What are the degrees of each of the graph's vertices?*

Problem 20 *Was it possible to design a Eulerian walk in the city of Königsberg at the time of Euler? Why or why not?*

Problem 21 *Find a Eulerian path in the following graph.*



Problem 22 *Does the above graph contain a Eulerian cycle? Why or why not?*

Problem 23 *What is the minimal number of times you need to lift the pencil off the paper to draw a cube without repeating any edge?*

6 Hamiltonian paths and cycles

A *Hamiltonian path* is a path in a graph that visits each vertex exactly once. A *Hamiltonian cycle* is a closed Hamiltonian path. They are named in honour of the great Irish mathematician, physicist, and astronomer, Sir William Rowan Hamilton (1805-1865).

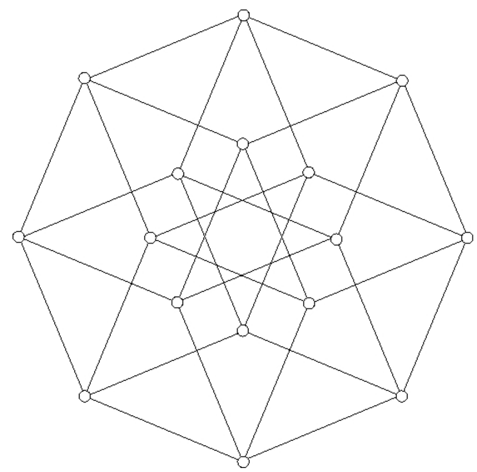
Sir William Rowan Hamilton

Problem 24 *A graph contains a Hamiltonian cycle. What is the minimal number of its vertices?*

Theorem 2 *(Ore) Let G be a connected graph with $n \geq 3$ vertices. If $deg(u) + deg(v) \geq n$ for every pair of non-adjacent vertices u and v, then G is Hamiltonian.*

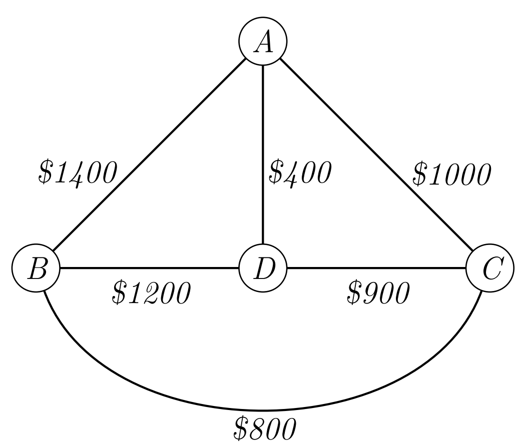
Problem 25

- *Does the graph below satisfy the conditions of Ore's theorem?*
- *Find a Hamiltonian cycle in the graph.*



Travelling salesman problem

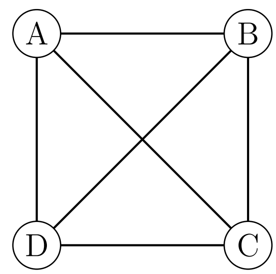
Problem 26 *A salesman with the home office in Albuquerque has to fly to Boston, Chicago, and Denver, visiting each city once, and then to come back to the home office. The airfare prices, shown on the graph below, do not depend on the direction of the travel. Find the cheapest way.*



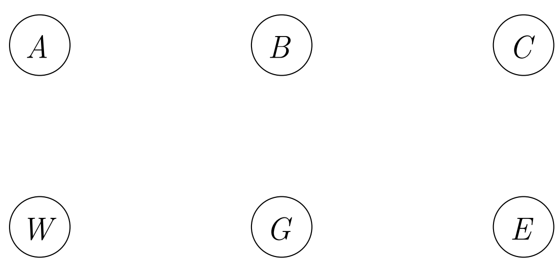
Problem 26 is a simple case of the *travelling salesman problem* (TSP). Let G be a graph, directed or undirected, with vertices $v_1, v_2, \ldots, v_n$. Its edges (v_i, v_j) are *weighted* – have numbers assigned to them. The TSP is to find a Hamiltonian cycle of lowest weight. The TSP appears in areas as different as scheduling, microchip design, DNA sequencing, and more.

7 Planar graphs

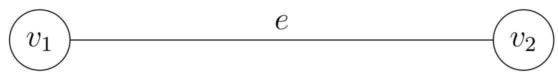
A graph is called *planar*, if it can be drawn in the plane in such a way that no edges cross one another. For example, the following graph is planar.

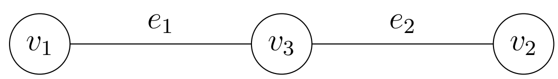


Problem 27 *Is it possible to connect three houses, A, B, and C, to three utility sources, water (W), gas (G), and electricity (E), without using the third dimension, either on the plane or sphere, so that the utility lines do not intersect?*



A *subdivision* of a graph G is a graph resulting from the subdivision of the edges of G. The subdivision of an edge $e = (v_1, v_2)$ is a graph containing one new vertex v_3, with the edges $e_1 = (v_1, v_3)$ and $e_2 = (v_3, v_2)$ replacing the edge e.

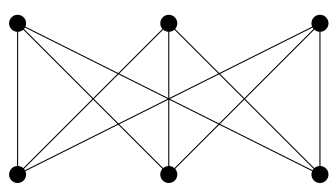




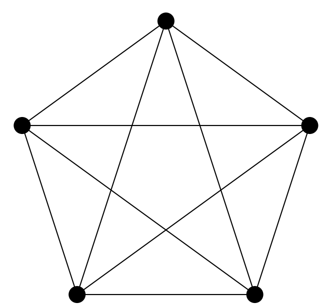
Problem 28 *What is the degree of a subdivision vertex?*

A graph H is called a *subgraph* of a graph G if the sets of vertices and edges of H are subsets of the sets of vertices and edges of G.

The following graphs are known as $K_{3,3}$ and K_5.



$K_{3,3}$



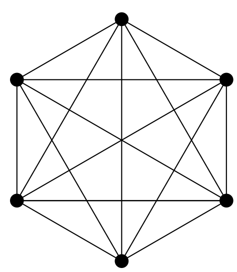
K_5

Let H be a graph that is a subdivision of either $K_{3,3}$ or K_5. If H is a subgraph of a graph G, then H is called a *Kuratowski subgraph*, after a famous Polish mathematician Kazimierz Kuratowski (1896-1980).

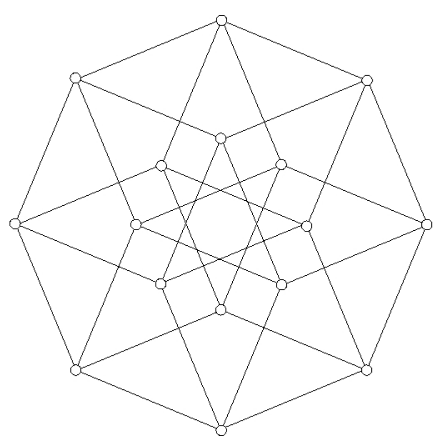
Kazimierz Kuratowski

Theorem 3 *(Kuratowski) A graph is planar if and only if it has no Kuratowski subgraph.*

Problem 29 *Is the following graph planar? Why or why not?*

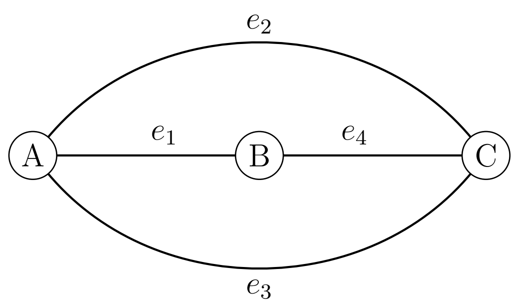


Problem 30 *Is the following graph planar? Why or why not?*



8 Euler characteristic

Let G be a planar graph, drawn with no edge intersections. The edges of G divide the plane into regions, called *faces*. The regions enclosed by the graph are called the *interior faces*. The region surrounding the graph is called the *exterior (or infinite) face*. The faces of G include both the interior faces and the exterior one. For example, the following graph has two interior faces, F_1, bounded by the edges e_1, e_2, e_4; and F_2, bounded by the edges e_1, e_3, e_4. Its exterior face, F_3, is bounded by the edges e_2, e_3.

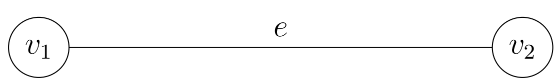


The *Euler characteristic* of a graph is the number of the graph's vertices minus the number of the edges plus the number of the faces.

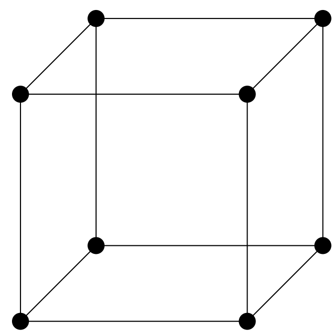
$$\chi = V - E + F \tag{1}$$

Problem 31 *Compute the Euler characteristic of the graph above.*

Problem 32 *Compute the Euler characteristic of the following graph.*



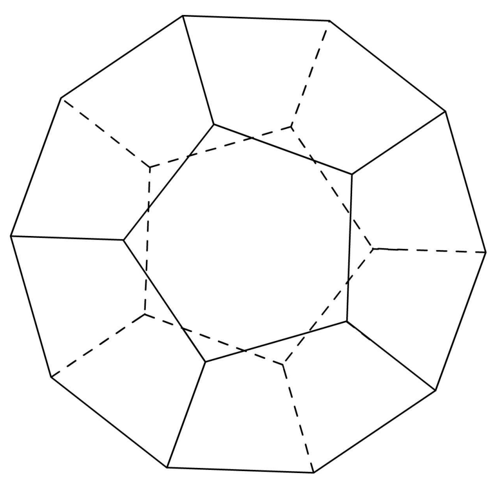
Problem 33 *Is the following graph planar? If you think it is, please re-draw the graph so that it has no intersecting edges. If you think the graph is not planar, please explain why.*



Problem 34 *Compute the Euler characteristic of the graph from Problem 33.*

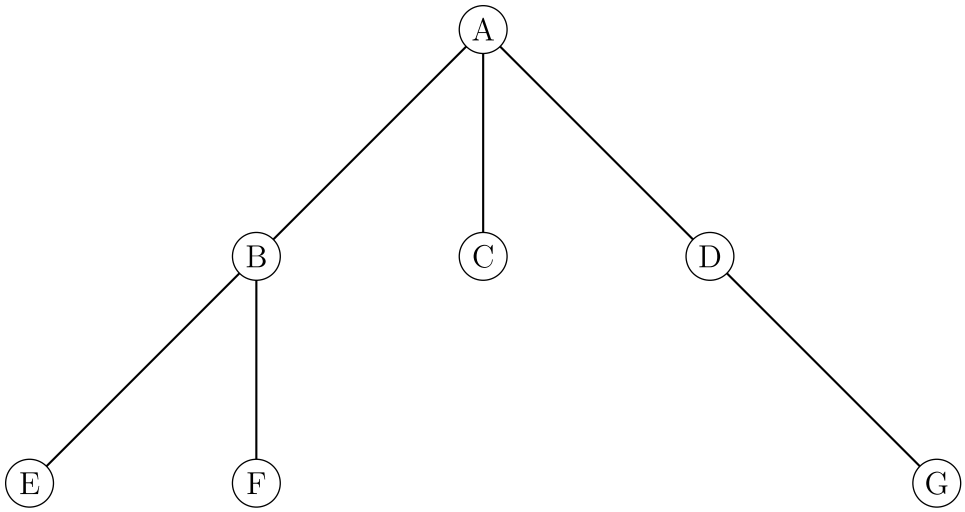
Let us consider the below picture of a *regular dodecahedron* as a graph, the vertices of the polytope representing those of the graph, the edges of the polytope, both solid and dashed, representing the edges of the graph.

Problem 35 *Is the graph planar? If you think it is planar, please re-draw the graph so that it has no intersecting edges. If you think the graph is not planar, please explain why.*



Problem 36 *Compute the Euler characteristic of the graph from Problem 35. Can you conjecture what the Euler characteristic of every planar graph is equal to?*

A graph is called a *tree* if it is connected and has no cycles. Here is an example.



A path is called *simple* if it does not include any of its edges more than once.

Problem 37 *Prove that a graph in which any two vertices are connected by one and only one simple path is a tree.*

Problem 38 *What is the Euler characteristic of a finite tree?*

Theorem 4 *Let a finite connected planar graph have V vertices, E edges, and F faces. Then $V - E + F = 2$.*

Problem 39 *Prove Theorem 4. Hint: removing an edge from a cycle does not change the number of vertices and reduces the number of edges and faces by one.*

Problem 40 *There are three ponds in a botanical garden, connected by ten non-intersecting brooks so that the ducks can sweem from any pond to any other. How many islands are there in the garden?*

Problem 41 *All the vertices of a finite graph have degree three. Prove that the graph has a cycle.*

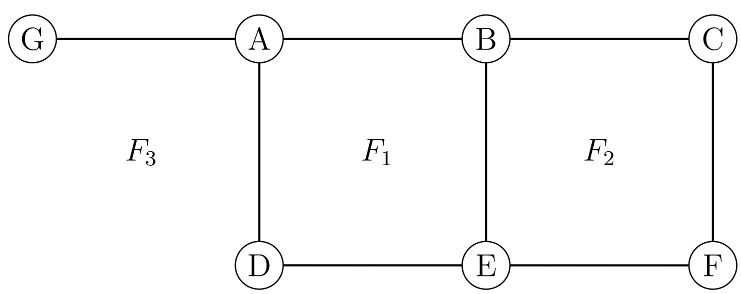
Problem 42 *Draw an infinite tree with every vertex of degree three.*

Problem 43 *Prove that a connected finite graph is a tree if and only if* $V = E + 1$.

Problem 44 *Give an example of a finite graph that is not a tree, but satisfies the relation* $V = E + 1$.

9 Proving that $K_{3,3}$ and K_5 are not planar

Let G be a planar graph with E edges. Let us call the *degree of its face*, $deg(F_i)$, the number of the edges one needs to traverse to get around the face F_i. For example, the following are the degrees of the faces of the graph below: $deg(F_1) = deg(F_2) = 4$, $deg(F_3) = 8$.



Note that in order to get around the exterior face of the graph, F_3, one has to traverse the edge $\{A, G\}$ twice.

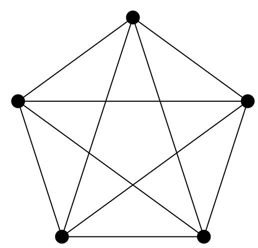
Problem 45 *Prove that* $\sum deg(F_i) = 2E$.

A graph is called *simple* if it is undirected, has no loops, and no multiple edges. The latter means that every pair of vertices connected by an edge is connected by only one edge. For example, the graph at the top of this page is simple, the graph at the top of page 21 is not.

Problem 46 *Let a finite connected simple planar graph have* $E > 1$ *edges and* F *faces. Prove that then* $2E \geq 3F$.

Problem 47 *Prove that for a finite connected simple planar graph,* $E \leq 3V - 6$.

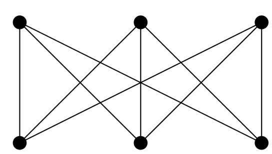
Problem 48 *Prove that the graph* K_5 *is not planar.*



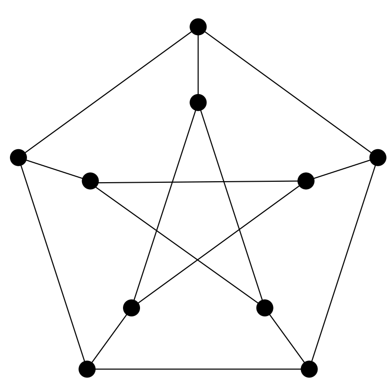
Problem 49 *Let* G *be a finite connected simple planar graph with* $E > 1$ *edges and no triangular faces. Prove that then* $E \geq 2F$.

Problem 50 *Let* G *be a finite connected simple planar graph with* $E > 1$ *edges and no triangular faces. Prove that then* $E \leq 2V - 4$.

Problem 51 *Prove that the graph* $K_{3,3}$ *is not planar.*

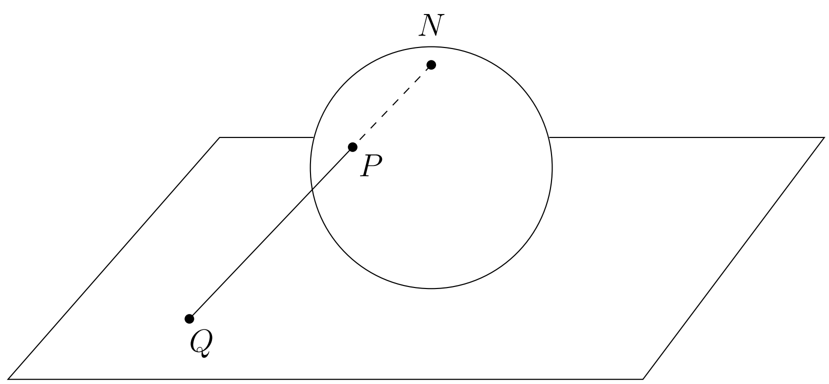


Problem 52 *The following graph is known as the Petersen graph. Is it planar? Why or why not?*



10 Stereographic projection

The following map from a sphere to a plane is known as the *stereographic projection*, $s(P) = Q$. (The sphere is tangent to the plane at the South pole.)



Properties of the stereographic projection

• Stereographic projection is *conformal*, i.e. preserves the angles at which curves cross one another.

• Stereographic projection does not preserve area.

• Circles on the sphere that do not pass through the North pole are projected to circles on the plane.

• Circles on the sphere that do pass through the North pole are projected to straight lines on the plane. These lines can be thought of as circles of infinite radius centered at infinity.

A (undirected) graph is called *complete*, if every pair of it distinct vertices is connected by a unique edge.

A *spherical tetrahedron* is a complete graph with four vertices on a sphere.

Problem 53 *Draw three spherical projections of the spherical tetrahedron on the Euclidean plane such that*

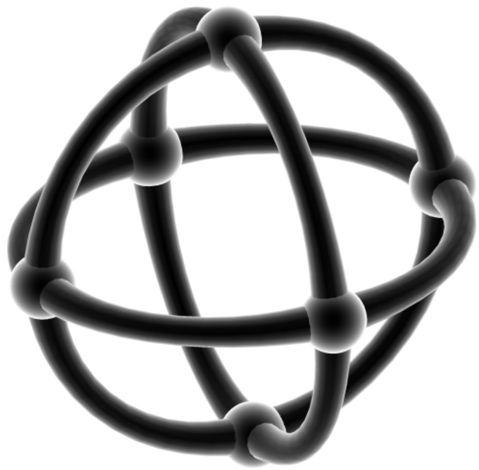
- *the North pole of the sphere coincides with one of the vertices of the original graph;*

- *the North pole of the sphere is an interior point of one of the edges of the original graph; and*

- *the North pole of the sphere is an interior point of one of the faces of the original graph.*

Note that a planar graph is a stereographic projection of a graph having no intersecting edges on a sphere. This observation sheds light on the somewhat mysterious infinite face of a planar graph. Its pre-image on the sphere is a finite face, like all others!

The following graph on a sphere is called a *spherical octahedron*.



Problem 54 *Draw a stereographic projection of the spherical octahedron such that the North pole of the sphere is an interior point of one of the faces of the original graph.*

Problem 55 *Find the Euler characteristic of a spherical octahedron.*

11 Platonic solids

A geometric figure B is called *convex* if for any two points P and Q of the figure, all the points of the straight line segment PQ are the points of B as well. A geometric figure that is not convex is called *concave*.

Problem 56 *Is a regular hexagon convex or concave?*

Problem 57 *Is any hexagon convex or concave?*

Problem 58 *One day, Sam drew two similar hexagons on a paper sheet and cut them out with scissors. Sam was quite surprised to find out that the larger hexagon never completely covered the smaller one no matter how he moved the figures on the table. Draw a pair of similar hexagons that have this property.*

Problem 59 *Is a cube a convex 3D solid?*

Problem 60 *Give an example of a convex 3D solid different from a cube.*

For a 3D polytope with V vertices, E edges, and F faces, let us call the number

$$\chi = V - E + F$$

its Euler characteristic.

Problem 61 *Prove that the Euler characteristic of any convex 3D polytope is equal to two.*

We have proven in Problem 47 that for a finite connected simple planar graph, $E \leq 3V - 6$. Please use this statement to prove the following.

Problem 62 *A connected simple planar graph contains at least one vertex of degree 5 or less.*

A *Platonic solid* is a 3D regular convex polyhedron. We have seen two in this course, a 3D cube and a regular dodecahedron on page 22.

Theorem 5 *There exist five different types of Platonic solids, a regular tetrahedron, cube, octahedron, dodecahedron, and icosahedron.*

Let us call D_f the degree of a Platonic solid's face and let us call D_v the degree of its vertex.

Problem 63 *Prove that* $\dfrac{1}{D_f} + \dfrac{1}{D_v} > \dfrac{1}{2}$.

Problem 64 *Prove that* $\dfrac{1}{D_f} > \dfrac{1}{6}$ *and* $\dfrac{1}{D_v} > \dfrac{1}{6}$. *Hint: use Problem 63 and the fact that* $D_f \geq 3$ *and* $D_v \geq 3$ *for a simple connected planar graph.*

Corollary (of Problem 64)

$$3 \leq D_v < 6 \qquad\qquad 3 \leq D_v < 6$$

Problem 65 *Does there exit a platonic solid with* $D_v = 3$ *and* $D_f = 3$*? If so, what is it?*

Problem 66 *Does there exit a platonic solid with* $D_v = 3$ *and* $D_f = 4$*? If so, what is it?*

Problem 67 *Does there exit a platonic solid with* $D_v = 3$ *and* $D_f = 5$*? If so, what is it?*

Problem 68 *Does there exit a platonic solid with* $D_v = 4$ *and* $D_f = 3$*? If so, what is it?*

Problem 69 *Does there exit a platonic solid with* $D_v = 4$ *and* $D_f = 4$*? Why or why not?*

Problem 70 *Does there exit a platonic solid with* $D_v = 4$ *and* $D_f = 5$*? Why or why not?*

Problem 71 *Does there exit a platonic solid with* $D_v = 5$ *and* $D_f = 3$*? If so, what is it?*

Problem 72 *Does there exit a platonic solid with* $D_v = 5$ *and* $D_f = 4$*? Why or why not?*

Problem 73 *Does there exit a platonic solid with* $D_v = 5$ *and* $D_f = 5$*? Why or why not?*

The surface of a standard soccer ball is made of 12 regular (spherical) pentagons and 20 regular (spherical) hexagons.

Problem 74 *Can the surface of a soccer ball be made of regular (spherical) hexagons only? Why or why not?*

Problem 75 *Can the surface of a soccer ball be made out of regular (spherical) pentagons only? If so, how many?*

12 Ramsey theory

Let us call two people friends if they know each other. Let us call them strangers otherwise.

Problem 76 *There are six people in the room. Prove that there are either three friends or three strangers among them. Hint: consider a graph with vertices representing people, red edges representing friendships, and blue edges representing the absence thereof.*

Problem 77 *There are five people in the room. Would there be necessarily either three friends or three strangers among them? Why or why not?*

Problem 78 *What is the minimal number of people in the room such that there are necessarily either three friends or three strangers in there?*

The above number is called $R(3,3)$.

Question 1 *What is the meaning of the number $R(2,5)$?*

Problem 79 *Find $R(2,5)$.*

Problem 80 *Is $R(r,b) = R(b,r)$ for any $r, b \in \mathbb{N}$? Why or why not?*

In general, the minimal number of vertices a complete graph must have in order to contain either a red subgraph with r vertices or a blue subgraph with b vertices is called the *Ramsey number* $R(r,b)$.

Theorem 6 (Ramsey) *The number $R(r,b)$ exists for any $r, b \in \mathbb{N}$.*

Frank Plumpton Ramsey (1903-1930), a British philosopher, mathematician, and economist.

Philosophical observation: pick up two natural numbers r and b. Take $R(r, b)$ or more points. Connect each of the points to all others, choosing one of the two different colors, red or blue, at random. What you get seems to be totally chaotic. However, one will always be able to find a monochromatic subgraph, either a red one with r vertices or a blue one with b vertices, within the original graph. Chaos generates order!

The last time we've checked, it was proven that

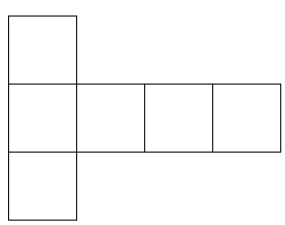
$$43 \leq R(5, 5) \leq 49,$$

but the exact value was not known.

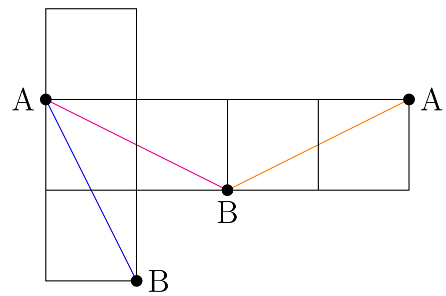
Problem 81 *Find $R(5, 5)$.*

13 Answers and solutions

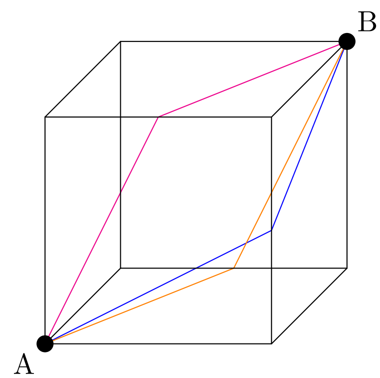
Problem 2 For example, this one.



Problem 3 The problem is easy to solve on a cubic net.



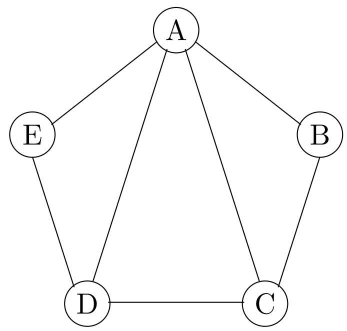
Folding the net back into the cube, we get the following three shortest paths.



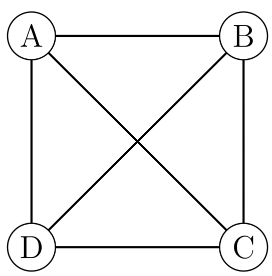
There exist three more. Point A is a vertex of three of the cube's faces. Each of the faces has two sides opposite to A. The ant can crawl to the center of either side and proceed to point B from there. The net used above has shown us three such paths. A different net would show others.

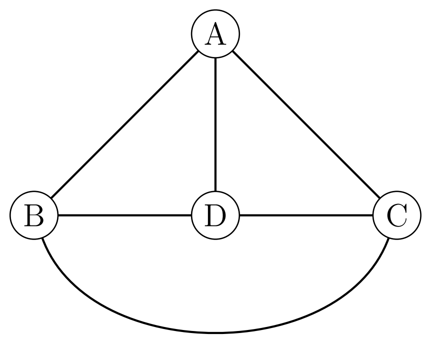
Problem 4 An edge $e = \{A, B\}$ that is not a loop has two different orientations, either (A, B) or (B, A). One can choose an orientation of each of the E edges independently from others. Therefore, there are 2^E different orientations.

Problem 5



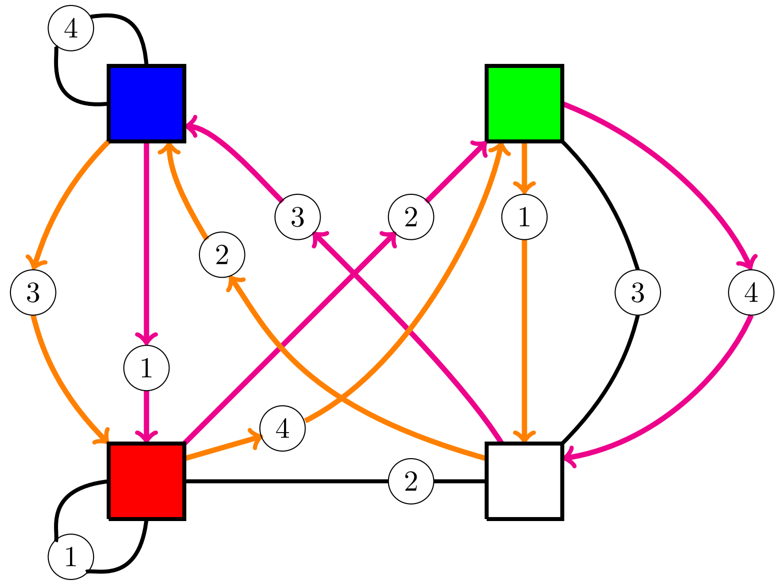
Problem 6 Let us call $\mathcal{V}_1$ the set of the vertices of the first graph and let us call $\mathcal{V}_2$ the set of the vertices of the second.





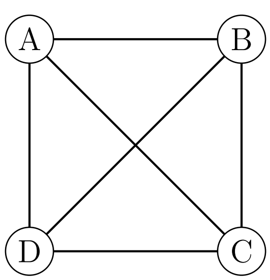
The sets are obviously equal, $\mathcal{V}_1 = \{A, B, C, D\} = \mathcal{V}_2$. Similarly, $\mathcal{E}_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\} = \mathcal{E}_2$. Therefore, the two pictures represent one and the same graph.

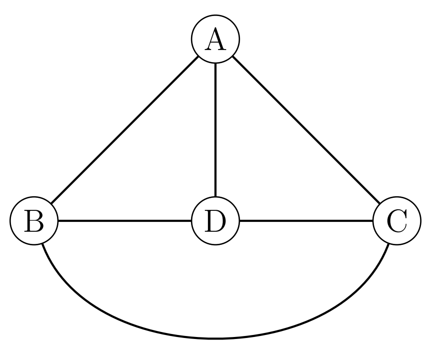
Problem 8 Rotate the cubes, keeping the front and rear sides in their current configuration, so that the colors of the cubes' upper and lower faces follow the orange oriented cycle on the graph below.



Problem 9 Each edge of an undirected graph is incident to exactly two vertices, the ones it connects.

Problem 10 Let $e_1 = \{A, B\}$, $e_2 = \{A, C\}$, $e_3 = \{A, D\}$, $e_4 = \{B, C\}$, $e_5 = \{B, D\}$, and $e_6 = \{C, D\}$ for each of the graphs.



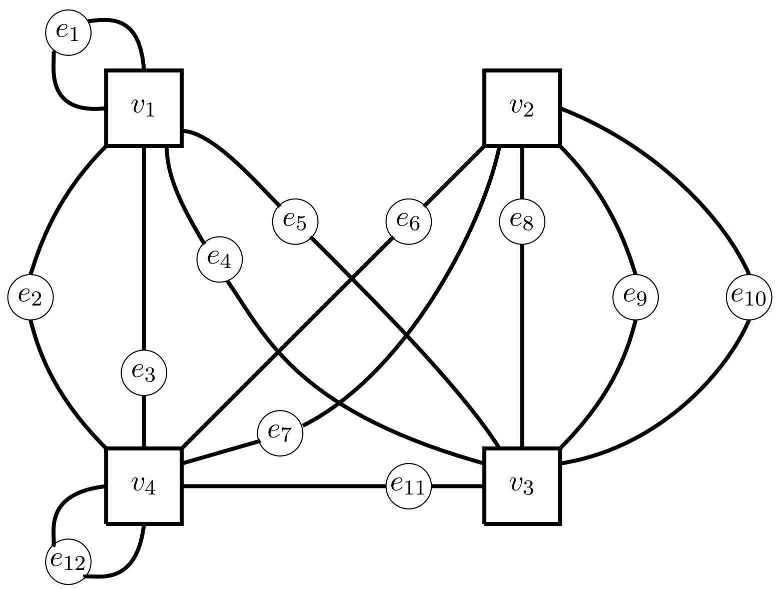


Then

$$T_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} = T_2.$$

Since an incidence matrix completely describes the structure of its graph, the fact that $T_1 = T_2$ proves that the two pictures above represent one and the same graph.

Problem 11 Let us number the vertices and edges on the graph as shown on the picture below.



Then the incidence matrix of the graph is the following.

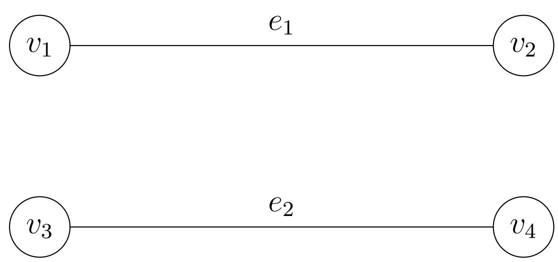
$$T = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Problem 13 Here is one way to count the edges of a graph. Go through the graph's vertices one-by-one, count the edges sticking out of each vertex and sum up the numbers. Note that this way you count each edge twice. Indeed, an edge connects two vertices and thus is sticking out of both of them! Therefore, the sum of the degrees of the vertices equals twice the number of the edges, Q.E.D.

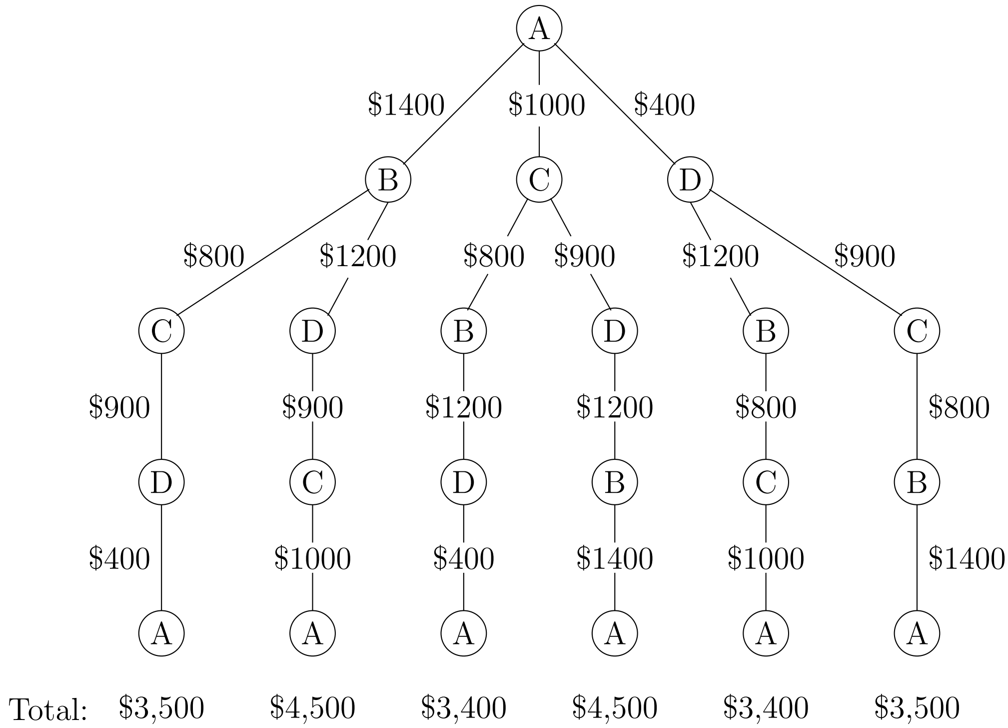
Problem 14 According to Theorem 1, the sum of the degrees of all the vertices of a finite graph is even. Therefore, the number of the vertices of odd degree in the graph must be even, too.

Problem 15 Let us represent kids as vertices of a graph. Let us connect two vertices by an edge if and only if the two corresponding children are friends. Then, according to the first girl, we should end up with a graph having an odd number of vertices (25), all of odd degree (5). However, due to Problem 14, the number of the vertices of odd degree must be even!

Problem 17 Draw a graph with four vertices, all of degree one, in the space below.

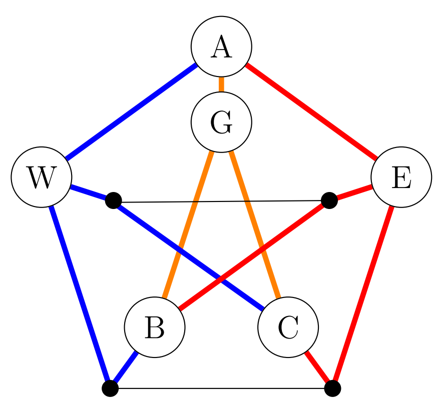


Problem 26 All the possible travel routes are described by a *weighted choice tree*, the three that branches at every available destination choice, with prices of flights assigned to the edges.

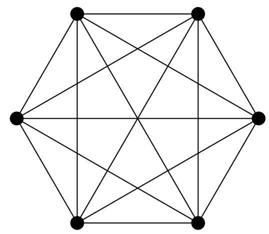


Adding up the weights along the branches, we find the total prices of all the trips. It turns out that the routes ACBDA and ADBCA are the cheapest, each at the price of $3,400.

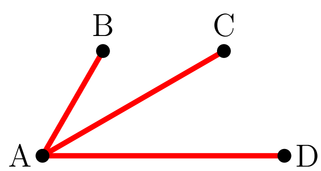
Problem 52 The Petersen graph is not planar because it has a Kuratowski subgraph, a subdivision of $K_{3,3}$, please see the picture below.



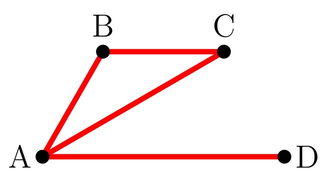
Problem 76 The graph to consider is K_6, a complete graph with six vertices.



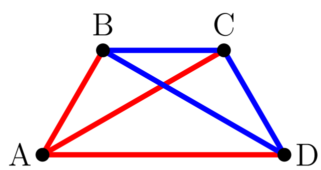
Each edge of the graph is colored red or blue. Since each vertex is incident to five edges, at least three edges incident to one and the same vertex must have the same color, for example, red.



Suppose one of the edges $\{B, C\}$, $\{B, D\}$, or $\{C, D\}$, for example, $\{B, C\}$, is red.

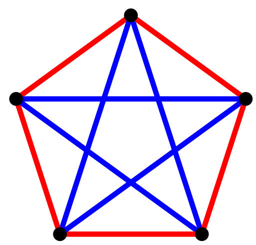


Then there are three friends in the room. Suppose that none of the edges $\{B, C\}$, $\{B, D\}$, or $\{C, D\}$ is red.



In this case, there are three strangers in the room.

Problem 77 The following coloring of the graph K_5 shows if there are five people in the room, it is possible that there are neither three friends nor three strangers among them.



Problem 78 It follows from Problems 76 and 77 as well as from their solutions above that $R(3, 3) = 6$.