

Geometry

Where everything is made up and the points matter

Based on LAMC handouts by Po-Shen Loh and Alin Galatan

Math Circle

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Today we will be returning to the roots of mathematics. Euclidean geometry. Euclidean geometry has a lot of historical significance, is visual, can be made totally rigorous, and is a mainstay of math competition problems. As a result we would be remiss to have a year of math circle without talking about geometry at all. For this discussion, I'm going to assume that you already have a background knowledge of triangles. Before we get into the

1. I'm going to assume for this handout that you are already familiar with triangles, some of their properties, and have some practice proving that two triangles are equivalent, or at least similar. In order to refresh your memory answer the following (short) questions.
 - (a) What does it mean for two triangles to be similar? How is that different to two triangles being equivalent?

- (b) Let A be short for angle, and S short for side. Which of the following similarities imply that two triangles are equivalent? Answer True or False. If your answer is false, draw a counterexample. Ask an instructor to explain this question if it isn't clear.

Relation	SSS	SSA	SAS	SAA	ASA	AAA
T/F						

(c) True or false, if a triangle is isosceles, then two of its angles are the same.

(d) True or false, a triangles' interior angles *always* sum to 180 degrees.

(e) What's the definition of the orthocenter of a triangle?

2. Ok, now let's get into the meat of things. A wise woman once said that Russian geometry books were better, because they didn't draw the picture for you like American books. At risk of Russify this circle, I won't be providing you with many pictures. Also, try and use Geometry where ever possible for the rest of this handout. You might have to use a tinsey bit of algebra (e.g. $a + b = c \iff a = c - b$, or cross multiplication) but keep it minimal.

(a) Given two points in plane, can you find a circle which circumscribes¹ these two points? Is that circle unique?

¹If some number of points are circumscribed by a circle C , then each point lies on the perimeter of C .

(b) Same question, but with three points

(c) Same question, but four points.

(d) Here's a useful fact. Suppose that I have a circle \mathcal{C} with center O . If $A, B \in \mathcal{C}$ the angle $\angle AOB$ depends only on the length of the chord AB (and the radius of \mathcal{C}). Draw a picture (or two) to illustrate this point. Check with an instructor to make sure that you have the right idea. Bonus, there is a bit of ambiguity in this question, but it's not a problem. What's the ambiguity? Why isn't it a problem?

(e) Let $C \in \mathcal{C}$. Prove that $\angle AOB = 2\angle ACB$.

(f) A very useful corollary of the above result is that $A, B, C, D \in \mathcal{C}$, then $\angle ACB = \angle ADB$. Prove this easy, but useful, result.

3. Now let's get onto the meat of the handout.

(a) Prove Ptolemy's theorem. Let $ABCD$ be a cyclic quadrilateral (a quadrilateral that can be circumscribed by a circle). Let X be the intersection of \overline{AC} and \overline{BD} . Prove that $|\overline{AX}||\overline{XC}| = |\overline{BX}||\overline{XD}|$.

(b) Prove also that the above property is not just an if, but if and only if.

(c) Prove that a quadrilateral $ABCD$ is cyclic if and only if $\angle A + \angle C = 180$.

(d) Let ABC be a triangle, and let D, E, F be the feet of the altitudes from A, B, C respectively. How many cyclic quadrilaterals can you find?

(e) In the triangle ABC $\angle B$ is right. Let $ACDE$ be the square surrounding ABC . Further, let M be the midpoint of $ACDE$. Find $\angle MBC$ in terms of the angles of the original triangle.

(f) Let ABC be a triangle, and let D, E, F be the midpoints of AB, BC, AC respectively. Prove that the three circles which pass through A, D, F & B, E, D & C, F, E all intersect at a common point.

(g) Let ABC be an acute triangle. A circle with diameter AB intersects the altitude CC' at points M and N . Similarly, a circle with diameter AC intersects altitude BB' at Q and P . Prove that $MPNQ$ is a cyclic quadrilateral.

(h) Let ABC be a triangle, and let M be a point on the circle which circumscribes ABC . Let D, E and F be the feet of the perpendiculars BC, AB , and AC . Prove that D, E and F are colinear.

(i) Let $ABCD$ be a quadrilateral, with the property that the diagonals are perpendicular. Let P be the intersection of the diagonals. Further, let W, X, Y and Z be the reflections of P across $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$. Prove that $WXYZ$ is a cyclic quadrilateral.

4. In the event that you finish all of the above problems, I have more that I can produce upon request.