Hanoi, Counting and Sierpinski's triangle

Infinite complexity in finite area

Math Circle

December 5, 2017

1. Per usual we are going to be watching a video today, but we aren't going to be watching it right at the beginning. Before we start the video, let's play a game.
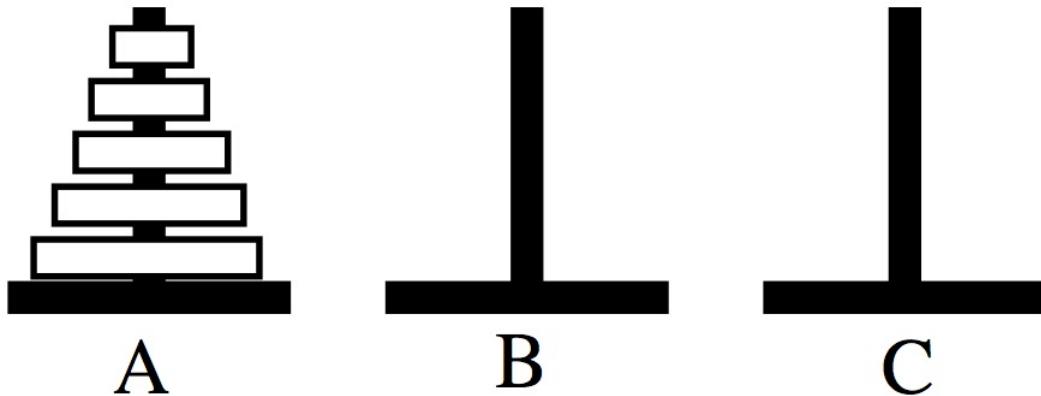

Figure 1: A schemetic for the Towers of Hanoi Puzzle. Taken from `http://www.cs.brandeis.edu/~storer/JimPuzzles/ZPAGES/zzzTowersOfHanoi.html`

The towers of Hanoi is a very simple game, here are the rules. Start with three wooden pegs, and five disks with holes drilled through the center. Place the disks on the first peg such that the radius of the disks is descending.

The goal of this game is to move all of the disks onto the very last peg so that they are all in the same order that they started. Here are the rules:

- You can only move 1 disk at a time from one peg to another.
- You can only move the top disk on a peg. If you want to move the bottom disk, you first have to move all of the other disks.
- You can never place a larger disk on top of a smaller peg. For example, if in the picture above if you moved the top disk of peg A onto peg B, you could not then move top new top disk from A into peg B.

The question is, is it possible to move all of the disks from peg A to peg C? If it is, what is the minimum number of moves that you have to do?

(a) Before we get into the mathematical analysis, let's try and get our hands dirty. Using the supplied materials, try and solve the Hanoi puzzle when you start with 3 pegs. For the paper version, use little bit of paper with a 1 of it represent a disk with radius 1, the bit of paper with a 2 on it represent a disk with radius 2, etc...

(b) You should be able to solve the 3 disk problem in 7 moves. Try and solve it in exactly 7 moves to convince yourself that it is possible.

(c) Next we are going to solve the same problem but this time starting with 4 pegs. **Before you solve it** make a guess as to how many move moves you will need. Think about it for a half of a minute, and write your guess down below. Don't worry about getting exactly the right answer, go with your gut.

(d) Now go ahead and actually try and solve the 4 disk puzzle.

(e) Make another guess, and then try and solve the 5 disk puzzle.

2. Ok, now let's move onto the regularly scheduled video for this week! This one if from a channel called 3Blue1Brown, and is called 'Binary, Hanoi and Sierpinski, part 1'

    https://www.youtube.com/watch?v=2SUvWfNJSsM

So that was cool, depending on your definition of cool. Let's go through his explanation and fill some of it in.

(a) First, let's make sure that everyone is on the same page with regard to different bases. We are used to counting in base 10, meaning that when you see the number:

$$2017$$

you interpret it as:

$$2 * 10^3 + 0 * 10^2 + 1 * 10^1 + 7 * 10^0$$

with this in mind, base two seems like a very reasonable number system as well. in base two, when you see:

$$10110$$

you are suppose to interpret it as

$$1 * 2^4 + 0 * 2^3 + 1 * 2^2 + 1 * 2^1 + 0 * 2^0.$$

Please rewrite the following base 2 numbers into base 10:

$$100 =$$
$$10000 =$$
$$10101 =$$
$$11111111 =$$

(b) Write down the numbers from 1 to 15 in order in their binary form please.

(c) One of the rules of the binary system is that you can only use the digits 0 and 1. Why is that? If we interpreted 1301 as a binary number we would get:

$$1 * 2^3 + 3 * 2^2 + 0 * 2^1 + 1 * 2^0 = 8 + 12 + 1 = 21$$

What's wrong with that? It doesn't look like anything wrong... Why does this rule exist? Is it just aesthetic? *Hint, 'good' counting systems have a useful property that is lost if you allow digits to be larger than their base. What is this useful property and why is it lost?*

(d) Most people can count to 5 on one hand. Using binary, you can actually represent a lot more than 5 digits using just the fingers on one hand. A fist is zero, the thumb extended only is 1, the pointer finger extended only is 2, the thumb and pointer finger is 3, etc... How many number could you represent with your digits on one hand in this way? What about both hands? *Hint, you have to be careful counting with this system, it can be very impolite! You don't want to accidentally '4' someone, if you know what I mean...*

(e) Did you know that there is such a thing as base 1 counting? It's called unary. Can you think about how you could represent all integers using 1 symbol only? Try and figure out what it is without looking it up.

(f) **The number $\phi = \frac{1+\sqrt{5}}{2}$ is often called the golden ratio. You can count things in the 'golden ratio base' using only the digits 0 and 1. It turns out once you know that $2 = \phi^1 + \phi^{-2}$ in this system (which is not hard to show), you can prove that every integer can be represented uniquely as a finite decimal of 0's and 1's in this system without having to have 2 adjacent one's in your representation. Can you prove this fact?**

Great news everyone, you get a bonus video today! Today we are also going to watch the part 2 of that video series

$$\texttt{https://www.youtube.com/watch?v=2SUvWfNJSsM}$$

3. One of the important and recurring themes in the two videos is the idea of recursion. This is the core link between counting, the Sierpinski triangle and the Tower of Hanoi is the idea of recursion.

The core of every recursive problem is to solve a given problem by instead solving a similar, slightly simpler problem, and use the answer to that simpler problem to answer the original problem. That's kind of vague, so let's make it more explicit.

(a) Let's return to the original Tower of Hanoi problem for a second, where you can move disks from any one peg to another. Suppose that we want to solve the problem of moving 3 disks from one peg to another. Let's call this problem M_3 where M is supposed to remind us of Moving, and 3 is the number of disks to move. Suppose that we somehow knew how to solve M_2. Then we could use M_2 to figure out M_3 by doing:

$$M_3 = M_2, D_3, M_2 \tag{1}$$

where the right side of that equation is supposed to be read 'Move two disks, move disk three, move two disks.' Using a similar technique, write down M_2 in terms of D_2 and M_1.

(b) Great, now what is M_1? It's just moving one disk, which is the same as D_1. Using backwards substitution, find out what M_2 is in terms of just D. What about M_3? We call D a 'base operation' because it's the simplest problem that we can solve without recursing again.

(c) How many operations does it take to apply M_3 in terms of M_2? What about M_4 in terms of M_3? What about M_i in terms of M_{i-1} where $i > 1$?

(d) This should remind you of induction. Can you remember what induction was?

(e) What is the connection between induction and recursion?

(f) Now let's think about counting in base 2 as a recursive operation. Suppose that the base operation is incrementing a number by 1. Suppose that C_2 was counting to 11, and we wanted to figure out how to count to $C_3 = 111$. Let's say that I is the increment operation. How can we write C_3 in terms of C_2 and I? *Hint, This is almost exactly the same thing as 3.a*

(g) Now let's return to the modified Hanoi problem where you can only move disks to adjacent pegs. Again, write M_3 in terms of M_2 and D_3.

(h) Now let's count in base 3, instead of base 2. Say that C_2 is counting 11 in base 3. How you write $C_3 = 111$ in terms of C_2 and I?

(i) Finally let's return to Sierpinski's triangle. Let's say that S_1 is a triangle, S_2 it a triangle with the middle quarter removed, S_3 is S_2 with the center fourth of each subtriangle removed, etc... Suppose that your base operation is drawing a single triangle (of any size). Write down S_3 in terms of S_2.

(j) **You should notice that this isn't quite the same thing as you might have expected. What gives? Why is this different from your answers to 3.g & 3.h?**

(k) **You have been selected to be a special effects specialist for the next Fast and Furious movie. In the movie there is a scene where the protagonist drives a futuristic looking car out the window of a 100 story skyscraper, lands on the floor and drives off. The problem is that if you drive it out of a floor too high up, then the car will be destroyed on impact, and you'll have to find a new actor to play the protagonist. If the car is driven out of a floor too close too the ground then the scene will be slightly less sweet than it could have been. Both of these outcomes are equally bad. Your mission is to find the highest floor that you can drive the car out of, without smushing the car, and it's driver.**

To help with this, you have two perfect (driverless) replicas of the car that you can use to test the fall before the main shoot. Once a car is crushed, you can't use it again. If it takes you 1 hour to get the car up to any floor and drive it off, what is the minimum number of hours that it will take you to figure out what floor the movie car should be driven off of?

(l) **What about if the skyscraper was 1000 stories tall, and you had 3 replica cars?**