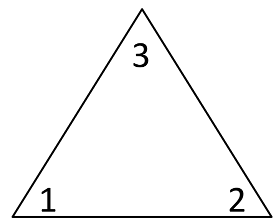
Junior Circle Meeting 8 – Geometric Transformations and Permutations

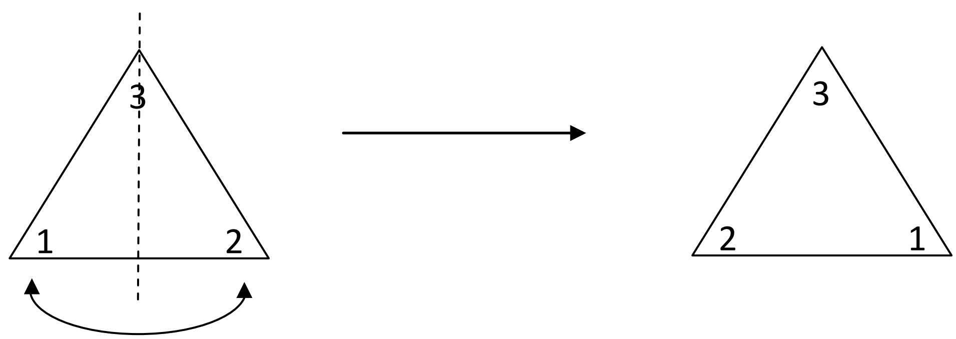
May 23, 2010

We are going to examine different ways to transform the triangle below:



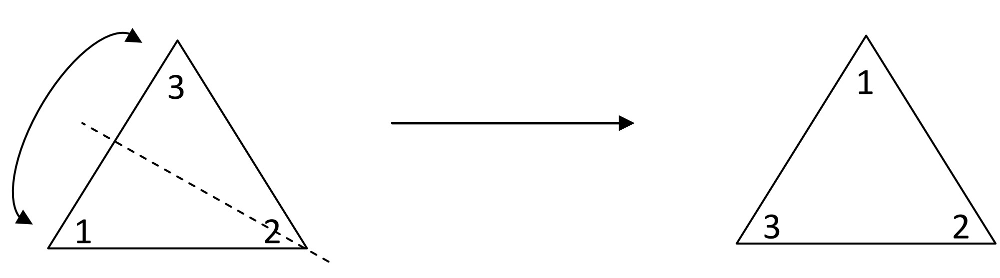
The first transformation is called a "flip". (This also can be called a "reflection.")

For example, we can flip the triangle across a line going through vertex 3.

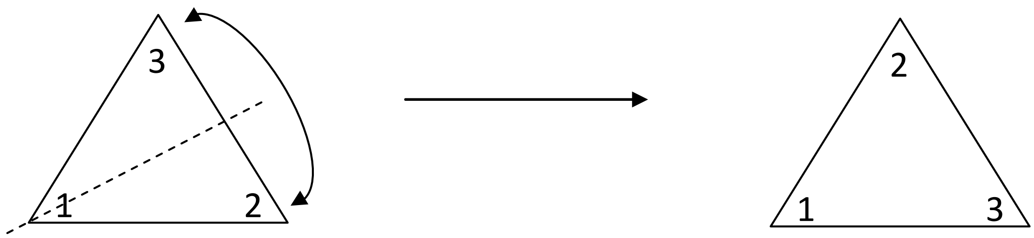


We call this flip F_3 because the axis of symmetry is the line going through vertex 3.

Similarly, we have F_2 :



And we have F_1 :



We can express each of these flips as a permutation. For example, for F_1 we write:

1	2	3
1	3	2

Meaning vertex 1 goes to vertex 1, vertex 2 goes to vertex 3, and vertex 3 goes to vertex 2.

1. Now we try this with the other flips!
   a. Describe F_2 as a permutation:

1	2	3

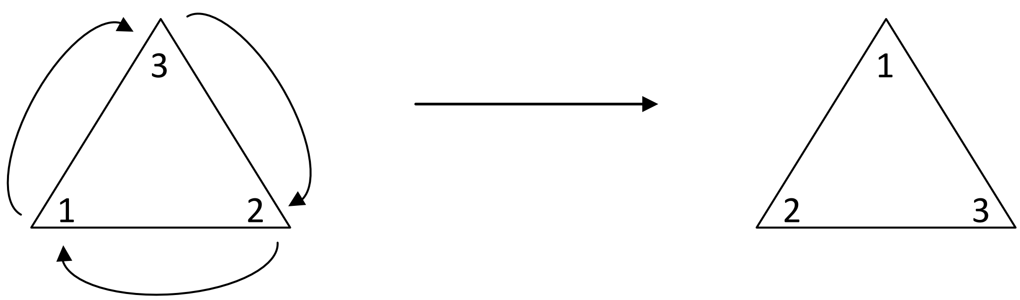
   b. Describe F_3 as a permutation:

1	2	3

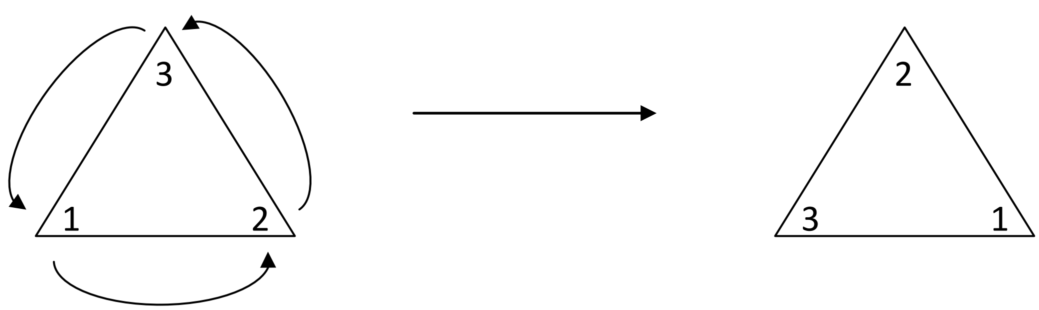
c. What do you notice about all three of these permutations? How many vertices change? How many vertices stay the same?

The second transformation is called a “rotation”.

For example, we can rotate the triangle clockwise. We call this $\overset{\curvearrowright}{R}$:



We also can rotate it counterclockwise. We call this $\overset{\curvearrowleft}{R}$:



2. Now we write the permutations corresponding to the rotations:

   a. Describe $\overset{\curvearrowright}{R}$ as a permutation:

1	2	3

   b. Describe $\overset{\curvearrowleft}{R}$ as a permutation:

1	2	3

   c. What do you notice about both of these permutations? How many vertices change? How many vertices stay the same?

3. Mindy remembers that there are 6 different three-letter words with the letters A, B, and C (using each letter only once).

   a. Mindy says that we can think of the vertices 1, 2, and 3 in the same way. Write down all three-digit numbers using 1, 2, and 3 (using each digit only once).

b. How many three-digit numbers did you get?

c. Mindy says that we can express each of these three-digits numbers as a permutation, so that we can write the three numbers in the empty boxes below:

1	2	3

1	2	3

1	2	3

1	2	3

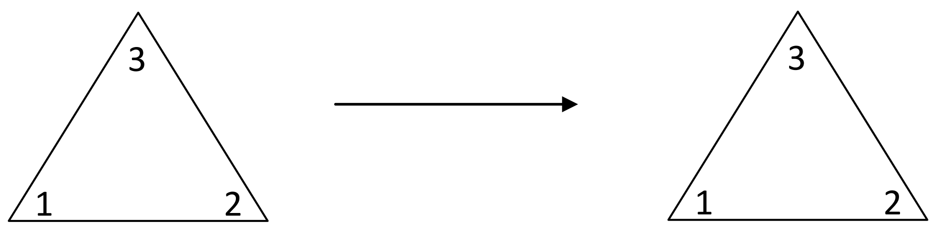
1	2	3

1	2	3

d. Which permutations above are flips? Which are rotations? Label the permutations you know as F_1, F_2, F_3, $\overset{\curvearrowright}{R}$, and $\overset{\curvearrowleft}{R}$.

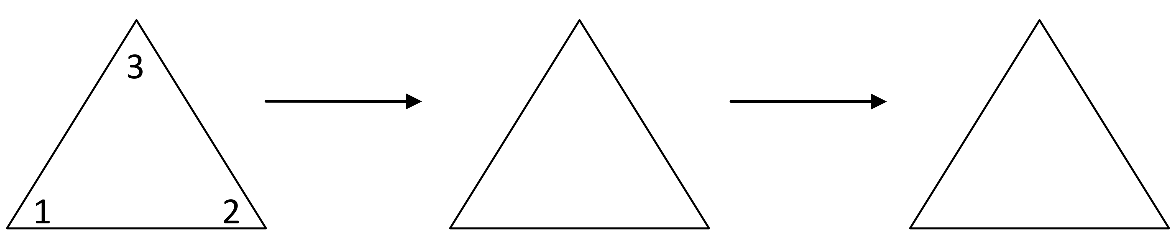
e. Mindy notices that there are 6 permutations above, but we only have 3 flips and 2 rotations. How can you describe what the 6th permutation does?

We call this final transformation the "identity" transformation and call it I :

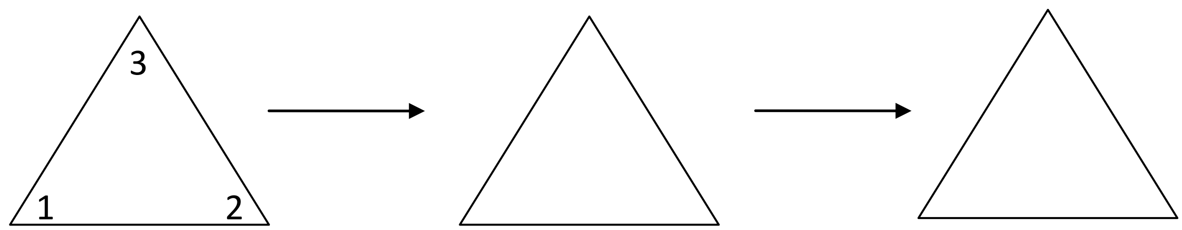


We now examine performing two or more of these transformations in a row. This is called a "composition" of transformations. Here, when we write a composition, we go from left to right. So for example, $F_1 \circ F_2$ means we first do F_1 and then do F_2 .

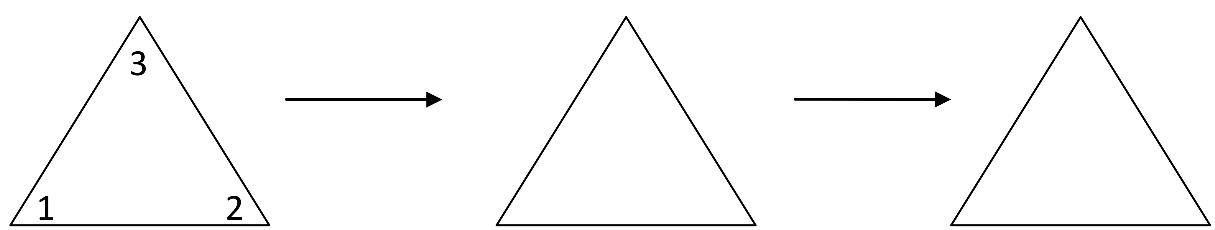
4. Use the triangles below to find the compositions.
   a. Which transformation is the same as $\overset{\curvearrowright}{R} \circ \overset{\curvearrowright}{R}$?



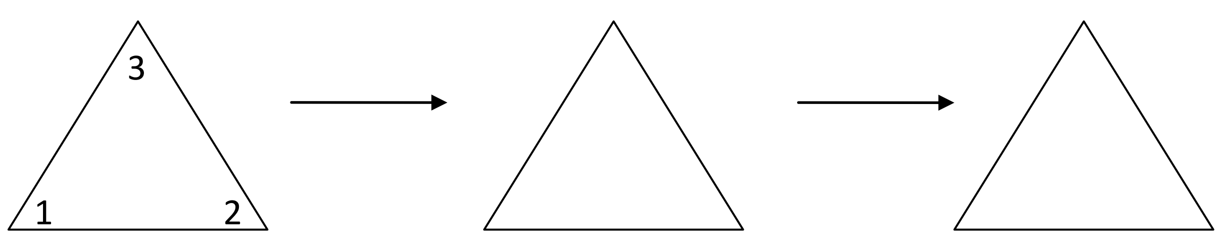
b. Which transformation is the same as $\overset{\curvearrowleft}{R} \circ \overset{\curvearrowleft}{R}$?



c. Which transformation is the same as $\overset{\curvearrowright}{R} \circ \overset{\curvearrowleft}{R}$?



d. Which transformation is the same as $\overset{\curvearrowleft}{R} \circ \overset{\curvearrowright}{R}$?

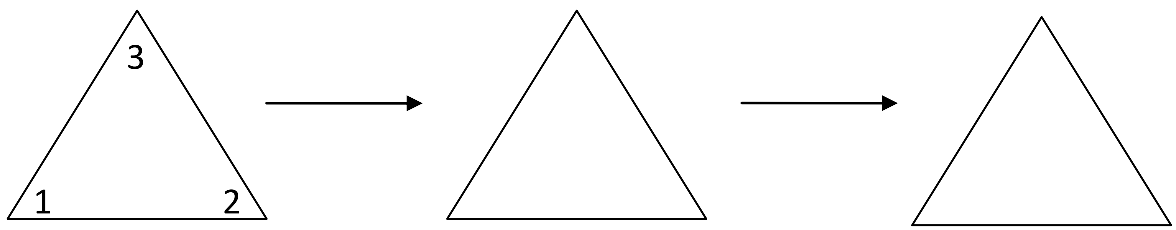


In parts (c.) and (d.) in problem 4 above, we notice that the order of the rotations does not matter. In other words, if we swap the order, we still get the same answer!

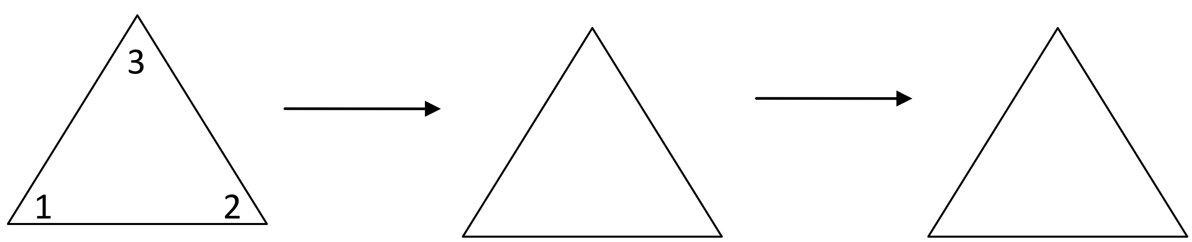
In this case, we say that the composition of the transformations *commute*.

Let's now find what other compositions of transformations commute!

5. Use the triangles below to find the compositions.
   a. Which transformation is the same as $F_1 \circ \circlearrowleft R$?

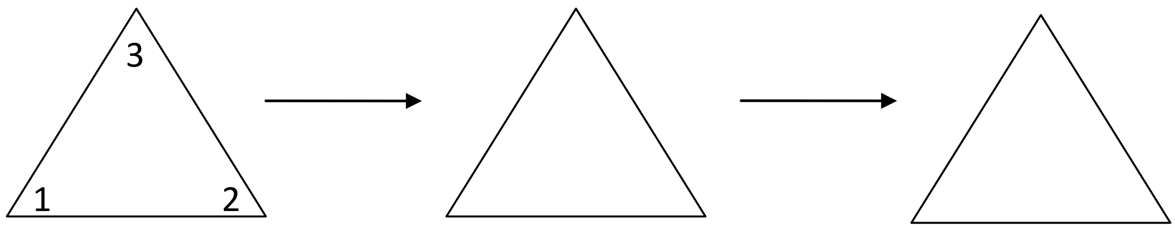


   b. Which transformation is the same as $\circlearrowleft R \circ F_1$?

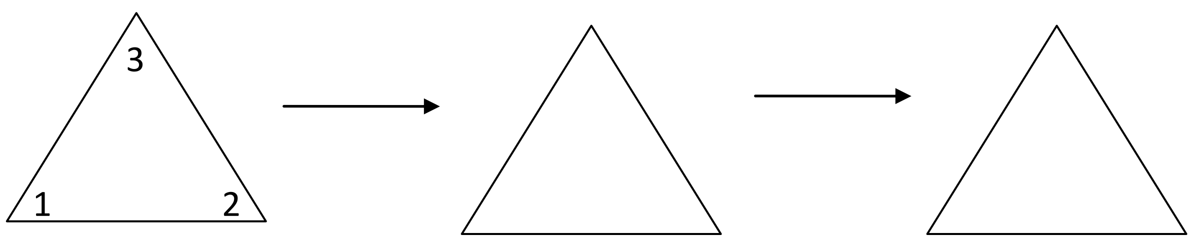


   c. Do F_1 and $\circlearrowleft R$ commute? In other words, do we get the same transformation in parts (a.) and (b.)?

6. Use the triangles below to find the compositions.
    a. Which transformation is the same as $F_1 \circ F_2$?



    b. Which transformation is the same as $F_2 \circ F_1$?

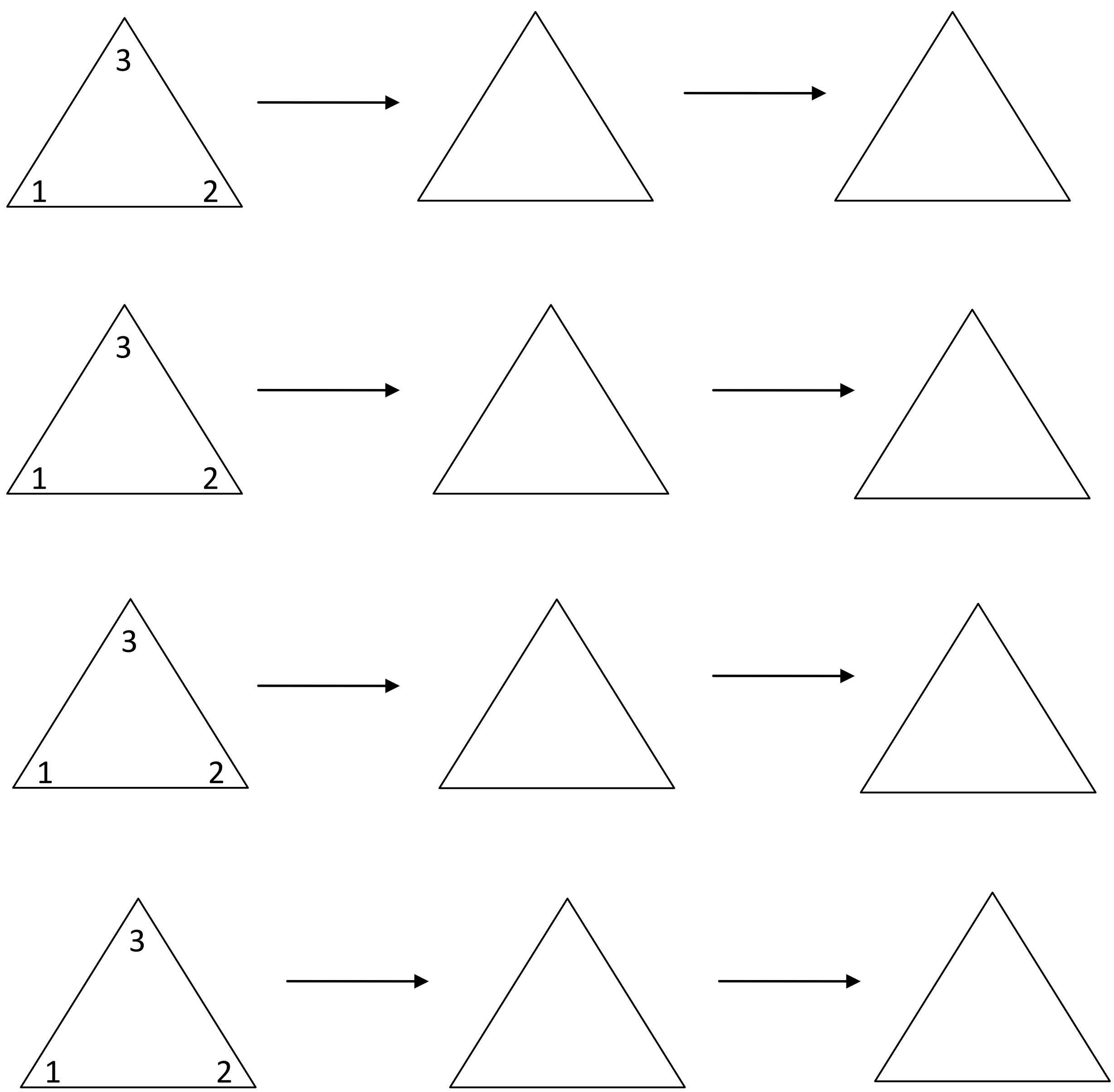


    c. Do F_1 and F_2 commute? In other words, do we get the same transformation in parts (a.) and (b.)?

7. Fill in the table below for the compositions of transformations. First do the operation in the left column, then do the operation in the top row:

↓First Second→	F_1	F_2	F_3	↺R	R↻
F_1					
F_2					
F_3					
↺R					
R↻					

You can use the triangles below to help you find these compositions:



8. Which pairs of transformations commute? Which ones do not commute?