

## Definitions

A **graph** is a set of vertices in which each pair of vertices either is or is not connected by an edge. (We will not consider graphs where there can be edges from a vertex to itself, or several distinct edges between two vertices.)

A **planar graph** is a graph that can be drawn in a plane without edge crossings. We say that a planar graph is **properly drawn** if it is drawn in this fashion. When properly drawn, a planar graph divides the plane into several regions (including the outside!), which we call **faces**.

A **walk** (from  $v$  to  $w$ ) is a sequence of vertices and edges  $(v =)v_0, e_1, v_1, e_2, \dots, e_n, v_n(= w)$ , where  $v_{i-1}, v_i$  are vertices connected by the edge  $e_i$ .

A **path** is a walk  $v_0, e_1, v_1, e_2, \dots, e_n, v_n$  where the vertices  $v_1, v_2, \dots, v_n$  are all distinct.

Two distinct vertices  $u, v$  are **adjacent** if there is an edge with ends  $u$  and  $v$ . In this case we let  $uv$  denote such an edge.

The **degree** of a vertex  $v$  is the number of edges connected to  $v$ .

A vertex is **odd** if its degree is odd, and **even** if its degree is even.

A graph is **connected** if for every pair of vertices  $v, w$  there is a walk from  $v$  to  $w$ .

A **cycle** is a walk from a vertex to itself which contains at least one edge.

A **tree** is a connected graph without cycles. A **forest** is a graph without cycles (which is not assumed to be connected).

The **complete graph** on  $n$  vertices is the graph which has  $n$  vertices, with every pair of vertices connected by an edge.

## Graph (In-)Equalities

**Euler's Formula:** For any connected planar graph,  $V - E + F = 2$ .

Each of the following problems gives an equation or inequality relating the number of vertices, edges, and faces ( $V$ ,  $E$ , and  $F$ ) of various types of graphs. Prove each statement. For each inequality, try to describe all cases where equality holds.

1. For any graph,  $E \leq \frac{1}{2}(V^2 - V)$ .
2. The sum of the degrees of all vertices is  $2E$ .
3. For a planar graph, the sum of the number of sides of all faces is  $2E$ .
4. For a planar graph with at least three vertices, (a)  $3F \leq 2E$  (b)  $E \leq 3V - 6$
5. For a planar graph where no face has fewer than four sides, (a)  $2F \leq E$  (b)  $E \leq 2V - 4$

## The Graph Reaper

Euler's formula and the other relationships above have some interesting consequences.

1. The number of odd vertices is even.
2. If a graph has 10 vertices, all of degree 5, then it is not planar.
3. The complete graph on 5 vertices is not planar.
4. One cannot hook up three houses to three utilities by lines without two lines crossing.
5. A planar graph must have some vertex whose degree is at most 5.
6. The vertices of any planar graph can be colored with five colors in such a way that no two vertices connected by an edge are the same color.

## Word Problems

1. There are 7 lakes in Lakeland, with some of the lakes connected by canals. If there are 10 canals and no two canals cross, how many islands are there?
2. There are 20 points inside a square, connected with straight lines to each other and to the vertices of the square in such a way that the square is divided into triangles. How many triangles are there?