

MATH CIRCLE HIGH SCHOOL 2 GROUP, WINTER 2018
WEEK 3: QUANDLES AND KNOT INVARIANTS

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1. QUANDLES

Definition. A Quandle is a set Q with two binary operations, \triangleright and \triangleleft (which I will pronounce "quandle" and "unquandle"), satisfying the following axioms:

- (1) *The quandle operation preserves identity: For any $x \in Q$, $x \triangleright x = x$.*
- (2) *The quandle and unquandle operations undo each other: For any $x, y \in Q$, $(x \triangleright y) \triangleleft x = y = x \triangleright (y \triangleleft x)$.*
- (3) *The quandle operation distributes over itself: For any $x, y, z \in Q$, $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$.*

A set with operations satisfying the latter two axioms is called a "Rack," while a set with an operation satisfying the last operation is called a "Shelf" - but these names are mostly historical oddities, as we'll only be considering quandles today.

Problem 1. *Prove that in any quandle Q , the following hold:*

- (1) *For any $y \in Q$, $y \triangleleft y = y$*

- (2) *For any $x, y, z \in Q$, $(z \triangleleft y) \triangleleft x = (z \triangleleft x) \triangleleft (y \triangleleft z)$*

- (3) *For any $a, b \in Q$, there is a unique $x \in Q$ such that $a \triangleright x = b$ (this fact is sometimes taken as a replacement for the second axiom, with the unquandle operation left unmentioned).*

Problem 2. *If we happen to be in a quandle Q that is also associative, so $a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright c$ for all $a, b, c \in Q$, what is the value of $a \triangleright b$?*

Problem 3. *Find all the different quandles of with 1, 2, and 3 elements (where two quandles that are just renaming of each others elements are considered the same).*

Problem 4. *Prove that the following are examples of quandles:*

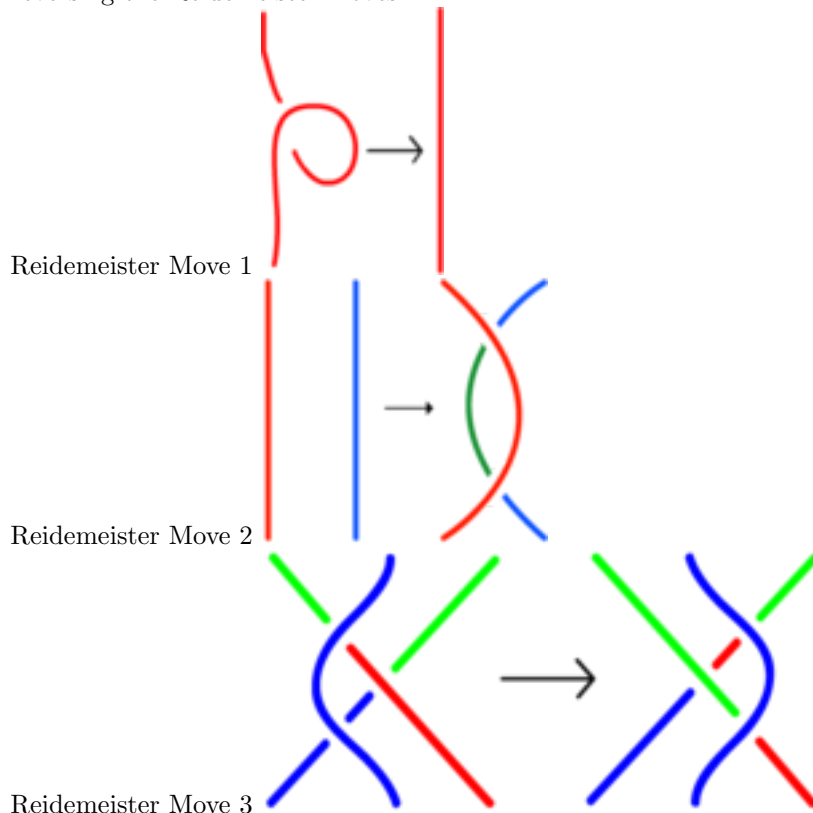
- (1) *The set of non-negative integers less than n , $\{0, 1, 2, \dots, n-1\}$ with the operation $a \triangleright b = 2a - b \pmod n$*

- (2) *The set of points in the plane, with $a \triangleright b$ being the point on the opposite side of a from b , the same distance away.*

- (3) *The set of points on the sphere, with $a \triangleright b$ being the point that b is sent to when the sphere rotates 180° around a .*

2. KNOT INVARIANTS

We have seen a bit of knot theory before, but here’s a reminder: a knot is a continuous closed curve in 3 dimensional space that is not allowed to intersect itself. In order to draw knots in two dimensions, we project them onto the plane, with the requirement that intersections only have two arcs hitting at a point, and always indicating which is on top. While two different pictures might represent the same knot, by applying or reversing the Reidemeister moves:

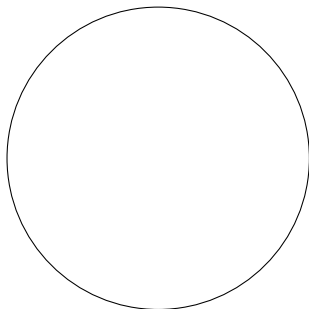
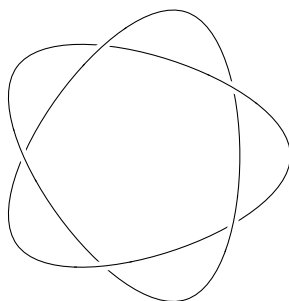
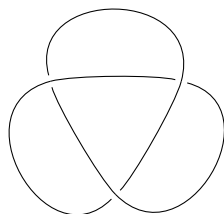
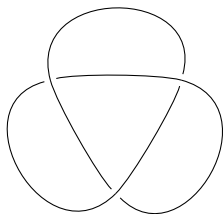


we can get between any two depictions of the same knot.

Definition. A knot invariant is something (like a number, or perhaps something more complicated) associated to a knot diagram, that does not change when any of the Reidemeister moves are applied.

Definition. A knot diagram is said to be tricolorable if you can assign one of three colors to each unbroken curve in the diagram so that each crossing has either 1 or 3, but never 2, distinct colors, and not all curves are given the same color.

Problem 5. Which of these knots are tricolorable?



Problem 6. Prove that tricolorability is an invariant — check that if a knot is tricolored, and you apply or reverse any of the Reidemeister moves, it can remain tricolored.

3. THE KNOT QUANDLE

Given any knot, it turns out that you can build a quandle!

Definition. *The knot quandle of a given knot diagram is constructed by labeling each of the arcs in the knot diagram, making sure to label on the same side of the curve as you follow it around. These labels generate the quandle; it is subject to relations obtained at each of the crossings. Assuming the top strand is labeled b and that label is on the righthand side of the strand (rotate the picture if necessary), and that the label on the left strand is a and the right strand is c : if the labels a and c are below their strands, we get the relation $a \triangleright b = c$, while if the labels a and c are above their strands, we get the relation $a = c \triangleleft b$.*

Problem 7. *Draw what the above relations mean at a crossing.*

Problem 8. *Determine the knot quandle for each of the knots from problem 5.*

Problem 9. *Prove that the knot quandle is a knot invariant.*