


Cool Induction Problems

Use induction to solve each of the following problems, which are cooler than other problems.

1. If n lines are drawn in a plane, and no two lines are parallel, how many regions do they separate the plane into?
2. A circle and a chord of that circle are drawn in a plane. Then a second circle, and chord of that circle, are added. Repeating this process, once there are n circles with chords drawn, prove that the regions in the plane divided off by the circles and chords can be colored with three colors in such a way that no two regions sharing some length of border are the same color.
3. $2n$ dots are placed around the outside of the circle. n of them are colored red and the remaining n are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the correct point.
4. Prove it possible to cover a $2^n \times 2^n$ grid with L tiles consisting of 3 squares that look like:

5. (**) A sphere is covered with some number of “caps” which are hemispheres. Prove that it is possible to choose four hemispheres, and remove all others, while still keeping the sphere covered. (Hint: Sometimes it is easier to prove a more general statement than the one given.)

Induction and Games

1. Prove by induction that in the game Survivor, where players alternate turns taking away 1, 2, 3, or 4 flags, all multiples of 5 are P-positions and all other numbers are N-positions.
2. Prove by induction that in the game where one can take 1, 3, or 4 flags, the P-positions are numbers congruent to 0 or 2 mod 7 (i.e. numbers which are a multiple of 7 or a multiple of 7 plus 2).

Induction and Formulas

Induction can be a good tool for proving difficult formulas. It isn't any help *coming up* with the formulas, but once you have the formula, you can use induction to prove it.

Prove each of the following formulas by induction:

1. $1 + 2 + \dots + n = \frac{n(n+1)}{2}$.
2. $1 \cdot 2 + 2 \cdot 3 + \dots + (n-1) \cdot n = \frac{n-1}{n}$.
3. $1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$.
4. Prove that $(1 - \frac{1}{4})(1 - \frac{1}{9}) \dots (1 - \frac{1}{n^2}) = \frac{n+1}{2n}$.