Nim Sum

The Nim sum of two numbers a, b is denoted by $a \oplus b$. We form the Nim sum as follows: Write the two numbers in binary, and add "without carry", i.e. in each digit,

$$0 \oplus 0 = 1 \oplus 1 = 0, \quad \text{and} \quad 0 \oplus 1 = 1 \oplus 0 = 1.$$

Example: Let's compute $5 \oplus 6$. We write:

$$\begin{array}{ccll} 5 & = & 101 & \text{in binary} \\ 6 & = & 110 & \text{in binary} \\ 5 \oplus 6 & = & 011 & \;=\; 3 \end{array}$$

More examples:

(a) $5 \oplus 3 = ?$ $\quad \begin{array}{cccc} & 1 & 0 & 1 \\ \oplus & & 1 & 1 \\ \hline & 1 & 1 & 0 \end{array}$ $\quad$ So $5 \oplus 3 = 6$.

(b) $6 \oplus 3 = ?$ $\quad \begin{array}{cccc} & 1 & 1 & 0 \\ \oplus & & 1 & 1 \\ \hline & 1 & 0 & 1 \end{array}$ $\quad$ So $6 \oplus 3 = 5$.

(c) $13 \oplus 15 = ?$ $\quad \begin{array}{ccccc} & 1 & 1 & 0 & 1 \\ \oplus & 1 & 1 & 1 & 1 \\ \hline & & & 1 & 0 \end{array}$ $\quad$ So $13 \oplus 15 = 2$.

Properties:

- $a \oplus b$ is another whole number.
- $a \oplus b = b \oplus a$ (commutative)
- $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associative)
- $a \oplus a = 0$ for any number a.
- Given a and c, there is one and only one number b such that $a \oplus b = c$.

Computing Nim Sums

Compute each of the following Nim sums:

1. $2 \oplus 3 =$

2. $2 \oplus 4 =$

3. $8 \oplus 7 =$

4. $15 \oplus 14 =$

5. $25 \oplus 31 =$

More Nim Sums

Complete the following Nim-sums:

1. (a) $5 \oplus (?) = 10$ (b) $12 \oplus (?) = 3$ (c) $10 \oplus (?) = 76$ (d) $511 \oplus (?) = 512$

2. (a) $(?) \oplus 2 = 7$ (b) $(?) \oplus 21 = 3$ (c) $(?) \oplus 40 = 46$ (d) $(?) \oplus 1025 = 133$

Nim Practice

Each of the following is an N-position for Nim. Find a move to a P-position. The first several positions are given in binary. For the second batch, you (might) need to calculate them!

1.

	(3,5,7)		(2,14,15)		(15,129,511)		(123,456,789)
(a)	11	(b)	10	(c)	1111	(d)	1111011
	101		1110		10000001		111001000
	111		1111		11111111		1100010101

2. (a) (4,7,12) (b) (9,15,1234567654321) (c) (3,9,11) (d) (7,17,21)

Analyzing Nim

Claim: Our claim is that (a, b, c) is a P-position for Nim precisely when $a \oplus b = c$.

In order to establish this claim, and complete our analysis of Nim, we need to establish the following facts:

(a) From every P-position, it is only possible to move to N-positions.

(b) From every N-position, it is possible to move to a P-position.

In other words, we need to prove the following facts:

(a) If $a \oplus b = c$, and one of a, b, c is decreased, then we no longer have $a \oplus b = c$.

(b) If $a \oplus b \neq c$, then it is possible to decrease one of the numbers a, b, c so that $a \oplus b = c$.

Prove away!